The Poincaré-extended **ab**-index

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joint with Joshua Maglione and Christian Stump
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Various Guises of Reflection Arrangements
March 15, 2023

Outline

1. Big Picture
2. *R*-labeled Posets and Generalized Descent Sets
3. The coefficients of the extended **ab**-index
4. Connection to the (ordinary) **ab**-index

The Big Picture

Arrangements of Hyperplanes

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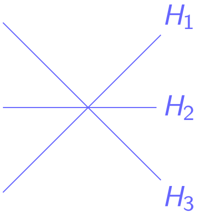
- A **hyperplane** is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.



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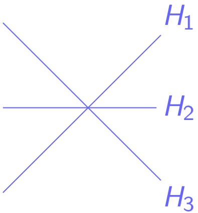

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- A **hyperplane** is an affine linear subspace of codimension 1.
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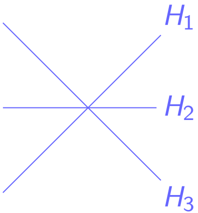
The set of intersections of this arrangement is

$$\mathbb{R}^2,\ H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$

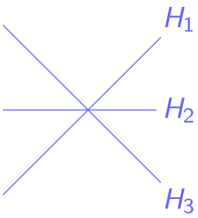


Today we'll focus on **intersections** (= nonempty intersections of some of the hyperplanes).

Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of **regions** of the arrangement.



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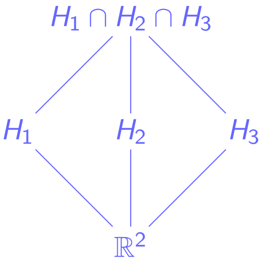

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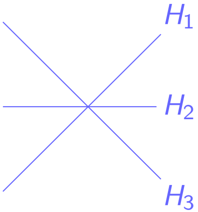

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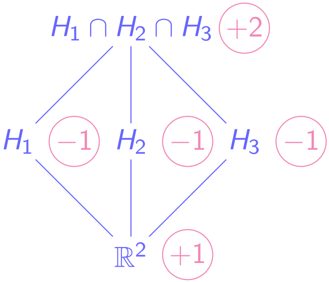

The Poincaré Polynomial of a Poset

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The **Poincaré polynomial** of $\mathcal{L}$ is

$$\mathrm{Poin}(\mathcal{L}; y) = \sum_{x \in \mathcal{L}} |\mu(\hat{0}, x)| \; y^{\mathrm{codim}(x)},$$

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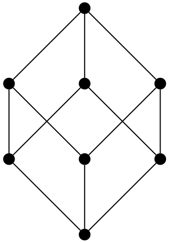
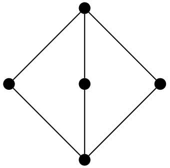
Tells us about the cohomology of the complement of the complexified arrangement, arises as the Hilbert series of $\mathfrak{gr}\, VG(\mathcal{A})$, and more!



$1 + 3y + 3y^2 + y^3$



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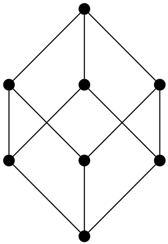
Note. We can define the Poincaré polynomial for any *graded poset*.



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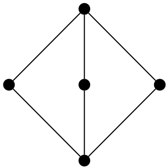

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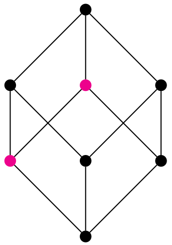
Chain Poincaré Polynomials

Let $\mathcal{A}$ be a central, essential hyperplane arrangement and $\mathcal{L}$ its lattice of intersections.

Moreover, let $\mathcal{C} = \{C_1 < \cdots < C_k\}$ be a chain of $\mathcal{L}$, i.e., a subset of the intersections which is totally ordered by inclusion.

The **chain Poincaré polynomial** of $\mathcal{C}$ is

$$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y) = \prod_{i=1}^{k} \mathrm{Poin}([C_i, C_{i+1}], y) \qquad \text{where } C_{k+1} = \hat{1}.$$



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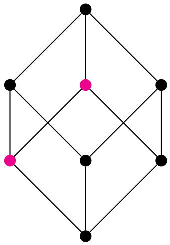

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Setting $y = 1$ recovers the size of a fiber of a chain under the *support map* $z : \Sigma^*(\mathcal{A}) \to \mathcal{L}$.

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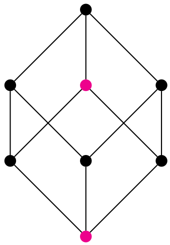

$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + 2y + y^2)(1 + y)$$

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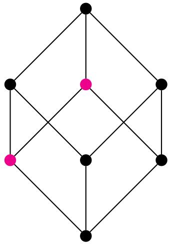
The Weight of a Chain

Let $\mathcal{A}$ be a central, essential hyperplane arrangement and $\mathcal{L}$ its lattice of intersections and let $\mathcal{C} = \{C_1 < \cdots < C_k\}$ be a chain of $\mathcal{L}$.

If P is rank n (every maximal chain from $\hat{0}$ to $\hat{1}$ has length $n+1$) then the **weight** of a chain $\mathcal{C}$ is $\text{wt}(\mathcal{C}) = w_1 \dots w_n \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ where

$$w_i = \begin{cases} \mathbf{b} & \text{if } \exists C_j \in \mathcal{C} \text{ such that } \text{rank}(C_j) = i - 1 \\ \mathbf{a} - \mathbf{b} & \text{else} . \end{cases}$$



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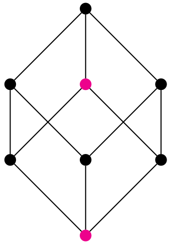

$$\mathrm{wt}(\mathcal{C}) = \mathbf{b}(\mathbf{a}-\mathbf{b})\mathbf{b}$$

The Poincaré-extended **ab**-index

Definition

The **(Poincaré-)extended ab-index** of $\mathcal{L}$ is

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } \mathcal{L}\setminus\{\hat{1}\}} \mathrm{Poin}(\mathcal{L}, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

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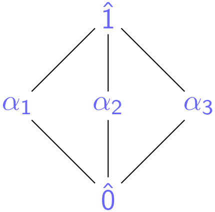

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+y)^2$	$\{0, 1\}$	$\mathbf{b}^2$

$$\begin{aligned}\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2\end{aligned}$$

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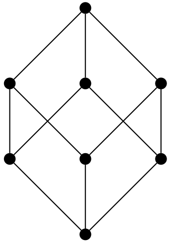

For the poset on the left:

$$\begin{aligned}\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \mathbf{a}^3 &+ (3y+2)\mathbf{a}^2\mathbf{b} + (3y^2+6y+2)\mathbf{aba} \\ &+ (3y^2+3y+1)\mathbf{ab}^2 + (y^3+3y^2+3y)\mathbf{ba}^2 \\ &+ (2y^3+6y^2+3y)\mathbf{bab} + (2y^3+3y^2)\mathbf{b}^2\mathbf{a} \\ &+ y^3\mathbf{b}^3\,.\end{aligned}$$

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(\mathcal{L}; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! Before we get into the proof, let's look at where their conjecture comes from...

Motivation: Analytic Zeta Functions

Let $\mathcal{A}$ be a central hyperplane arrangement in a real vector space with intersection lattice $\mathcal{L}$.

Maglione–Voll prove that (after a change of variables) the **(coarse) analytic zeta function** of $\mathcal{A}$ is

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \text{Poin}(\mathcal{C}\cup\{\hat{0}\}, y) \left(\frac{t}{1-t}\right)^{\#\mathcal{C}}.$$

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This is a bivariate version of the **analytic zeta function**.

A different bivariate specialization of their analytic zeta function recovers the celebrated **Motivic Zeta function** of a matroid given by Jensen–Kutler–Usatine.

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Putting all terms over the same denominator gives

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \frac{\mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}}{(1-t)^{\mathrm{rank}(\mathcal{A})}}.$$

The numerator of this rational function is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}.$$

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We can now state Maglione–Voll's conjecture more precisely:

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

Kühne–Maglione studied $Num_{\mathcal{A}}(1,t)$ as well, and conjectured that

$$\mathrm{Poin}(\mathcal{A},1)\cdot(1+t)^{\mathrm{rank}\mathcal{A}-1} \leq Num_{\mathcal{A}}(1,t).$$

We won't discuss it today, but our proof of the Maglione-Voll conjecture will give a proof of Kühne–Maglione's conjecture (almost) for free!

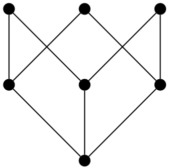
R-labeled Posets and Descent Sets

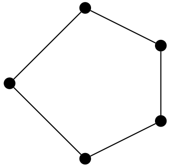
Graded Posets

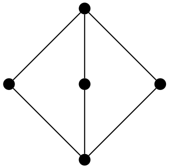
Let P be a poset with $\hat{0}$ and $\hat{1}$.

- A **chain** is a subset of the ground set which is totally ordered with respect to P.
- A chain $\mathcal{C} = C_1 < C_2 < \cdots C_n$ is **maximal** if C_i covers C_{i+1} for all $i = 1, \ldots, n-1$.
- P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.
- For $x, y \in P$, the **interval** between x and y is

$$[x, y] = \{z \mid x \leq z \leq y\}.$$





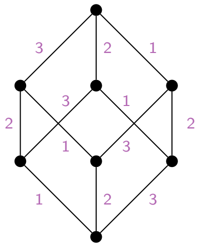


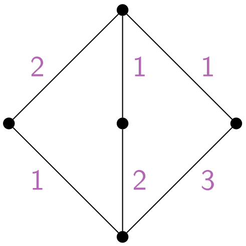
R-labelings

Let P be a graded poset, and let $\mathcal{E}(P) = \{(x, y) \mid x, y \in P,\ x \lessdot y\}$ denote the set of cover relations of P.

A labeling $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is an R-**labeling** if for every interval $[x, y]$, there is a unique maximal chain $\mathcal{M} = \{x = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = y\}$ such that the labels *weakly* increase, i.e.,

$$\lambda(C_{i-1}, C_i) \leq \lambda(C_i, C_{i+1}) \qquad \text{for } i = 2, \ldots k-1.$$



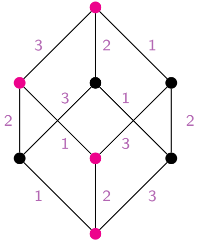


Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ.

Let $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ be a maximal chain of P. For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1}) .$$



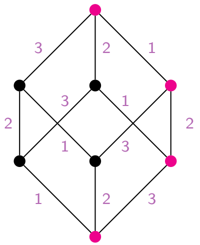
This chain has a descent at position 1.

Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ.

Let $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ be a maximal chain of P. For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1}) \,.$$



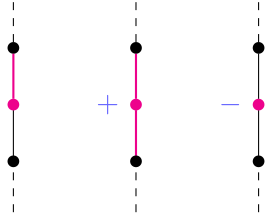
This chain has descents at positions 1 and 2.

Generalized Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ,

- $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ a maximal chain,
- E a subset of the edges of $\mathcal{M}$

For $i \in \{0, \ldots, n-1\}$, $(\mathcal{M}, E)$ has a **descent** at index i if we have one of the following situations

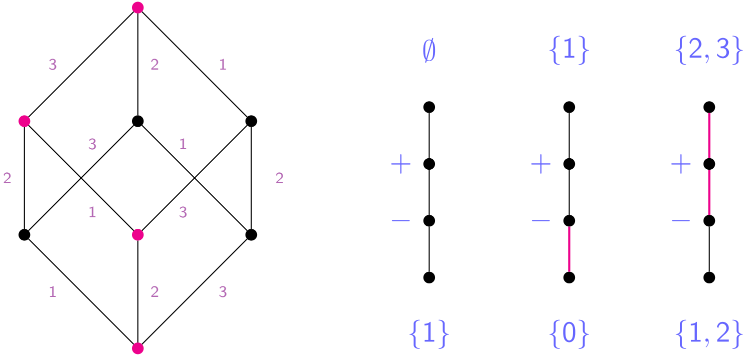


where $+$ means λ is increasing and $-$ means that λ is decreasing.

Now we include $i = 0$, which is a descent if and only if the edge above $\mathcal{M}_0$ is in E!

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.



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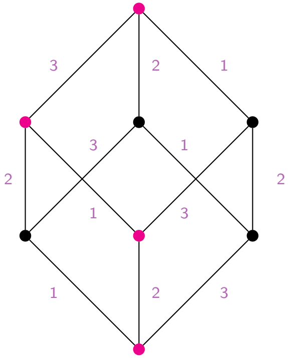
- E a subset of the edges of $\mathcal{M}$

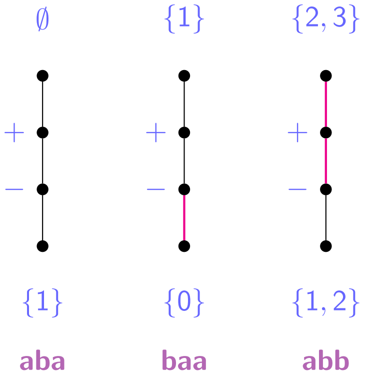
Then $\mathrm{mon}(M, E) = m_1 \dots m_n$ is the monomial in noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ with

$$m_i = \begin{cases} \mathbf{b} & \text{if } i \text{ is a descent of } (\mathcal{M}, E) \\ \mathbf{a} & \text{else}\,. \end{cases}$$

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets and monomials for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





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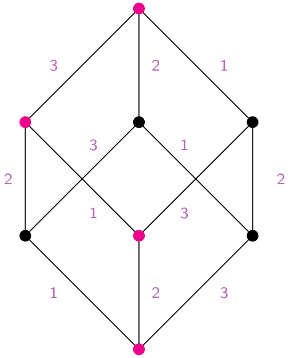
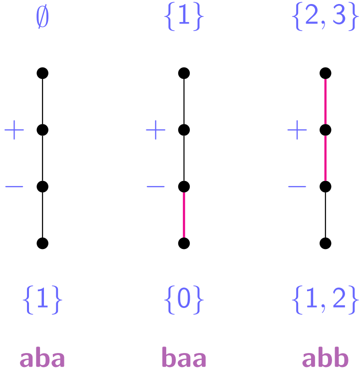




This descent statistic coincides with a statistic on *réseau* introduced by Bergeron, Mykytiuk, Sottile, and Willigenburg.

The coefficients of the extended **ab**-index

The Poincaré-extended **ab**-index

Let P be a graded poset.

Definition

The **extended ab-index** of P is

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(P; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})$, and holds for all posets with R-labelings!

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2023)

The extended **ab***-index of* P *is*

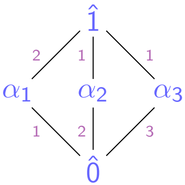
$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E}\mathrm{mon}(\mathcal{M}, E)$$

where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chain and* E *is a subset of its edges.*

This immediately implies a Maglione–Voll's conjecture.

Example

Computing $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ using the theorem instead of the definition.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$$

The Poincaré-extended **ab**-index

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Let's look at a short sketch of the proof...

(Overly-Simplified!) Proof Outline

Let P be a graded poset of rank n with an R-labeling λ.

Step 1: Use the following theorem to reinterpret the chain Poincaré polynomial as a sum over maximal chains with certain increasing-decreasing pattern with respect to the R-labeling.

Theorem

Let P be a poset with R-labeling λ. For $x, y \in P$ with $x < y$, we have

$$(-1)^{\text{rank}(x,y)}\mu(x,y) = \#\{\textit{decreasing maximal chains in } [x,y]\}.$$

Step 2: Use inclusion-exclusion to describe the coefficients as sets.

Step 3: Show that the elements at the top of this inclusion-exclusion argument are in bijection with pairs $(\mathcal{M}, E)$.

Connection to the (ordinary) **ab**-index

The (ordinary) **ab**-index

Definition

Let P be a graded poset. The **ab-index** of P is

$$\Psi(P;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P,\mathcal{C},0)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})\,.$$

The (ordinary) **ab**-index

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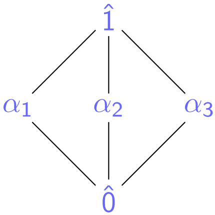

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L},\mathcal{C};0)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+0+0$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+0$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0}<\alpha_i\}$	$(1+0)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\Psi(\mathcal{L};y,\mathbf{a},\mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2+\mathbf{b}(\mathbf{a}-\mathbf{b})\\ &\quad +3\cdot(\mathbf{a}-\mathbf{b})\mathbf{b}+3\mathbf{b}^2\\ &= \mathbf{a}^2+2\mathbf{ab}\end{aligned}$$

The ω-map

Definition

Let m be a monomial in **a** and **b**. Define a transformation ω that first sends **ab** to $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}$, then all remaining **a**'s to $\mathbf{a} + y\mathbf{b}$ and all remaining **b**'s to $\mathbf{b} + y\mathbf{a}$.

If $\mathrm{m} = \mathbf{aabba}$, then

$$\omega(\mathrm{m}) = (\mathbf{a} + y\mathbf{b})(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb})(\mathbf{b} + y\mathbf{a})(\mathbf{a} + y\mathbf{b}).$$

By extending ω linearly, we can apply this map to sums of monomials, i.e.,

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= (\mathbf{a} + y\mathbf{b})(\mathbf{a} + y\mathbf{b}) + 2(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}) \\ &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}\,.\end{aligned}$$

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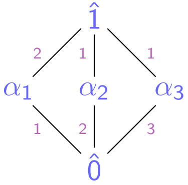
You might recognize these polynomials from earlier in this talk...

The ω-map

The **ab** index of the following poset is $\mathbf{aa} + 2\mathbf{ab}$.



We just saw that

$$\omega(\mathbf{aa} + 2\mathbf{ab}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}$$
$$= \mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})\,.$$

This is not a coincidence!

The ω-map

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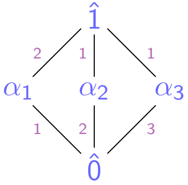

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This is not a coincidence!

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

The ω-map

Several specializations of the ω map have already appeared in the literature:

- When P is the lattice of flats of an *oriented matroid*, setting $y = 1$ recovers the ω map of Billera-Ehrenborg-Readdy,
- When P is the lattice of flats of an *oriented interval greedoid*, setting $y = 1$ recovers the ω map of Saliola-Thomas, and
- When P is a *distributive lattice*, setting $y = r + 1$ recovers the ω_r map of Ehrenborg (related to the "r-Signed Birkoff poset" from Hsiao).

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All three of these come from a pair of posets P, Q with an order- and rank- preserving surjection $z : P \to Q$ with the property that the size of the fiber $\#z^{-1}(\mathcal{C})$ of a chain $\mathcal{C}$ is an evaluation of $\mathrm{Poin}(Q, \mathcal{C}, y)$.

Future Questions

- There are posets not admitting R-labelings, which have nonnegative extended **ab**-indexes. What is this larger class of posets?
- What can we say about the coefficients of anayltic zeta functions themselves (can have negative coefficients)? What about the motivic zeta functions of JKU?
- The ω map can be reframed in terms of *peaks*. Setting $y = 1$ or $y = 0$ recovers well-studied combinatorics connected to *peak enumeration* and *quasisymmetric functions*. What can be said about y-refined peak enumerators?

Thank you for listening!

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