Arrangements & the Varchenko-Gelfand Ring

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joint with Nick Proudfoot, Jayden Wang, and Christian Stump

Texas A&M Algebraic Combinatorics Seminar
August 26, 2022

Outline

1. Hyperplane Arrangements & Open, Convex Sets
2. A Ring from Regions (arXiv 2208.04855)
3. Special Case: Catalan Numbers (arXiv 2204.05829)

Arrangements of Hyperplanes

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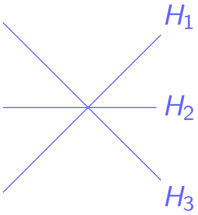
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.



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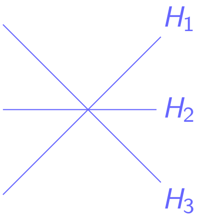

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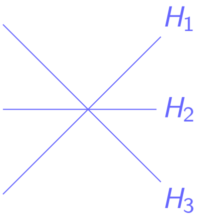
The following arrangement has 6 regions and the set of intersections is

$$\mathbb{R}^2, \; H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$



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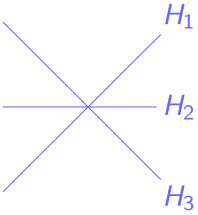
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of regions of the arrangement.



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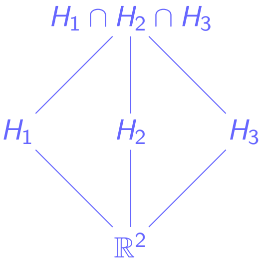

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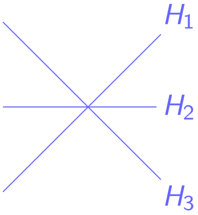

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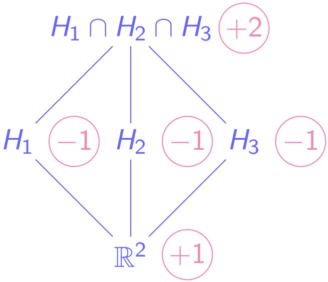

Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

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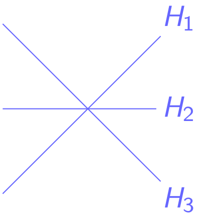
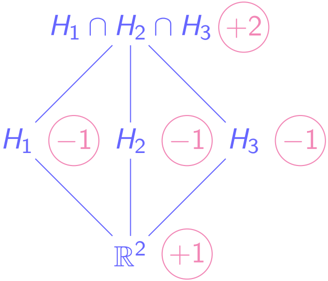




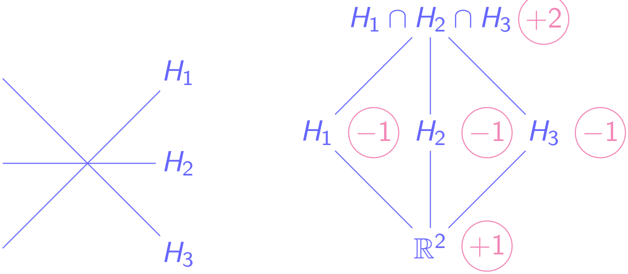
Zaslavsky's theorem says: $1 + 3(1) + 2 = 6$.

The Poincaré Polynomial

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$. Define the **Poincaré polynomial** of $\mathcal{A}$ by

$$\mathrm{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the arrangement.



The Poincaré polynomial of this arrangement is $\mathrm{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2$.

Hyperplane Arrangements and Open, Convex Sets

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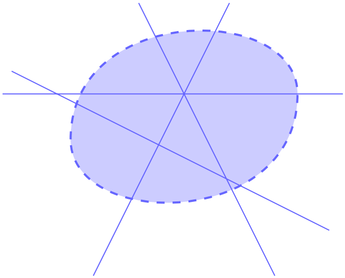
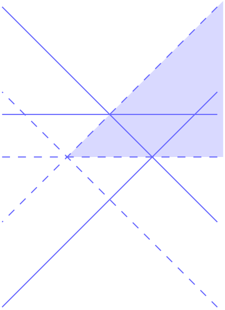
We will study the combinatorics of the pair $(\mathcal{A}, \mathcal{K})$.

Pairs $(\mathcal{A}, \mathcal{K})$ are interesting in the theory of arrangements, as they unify the theory of **central** and **affine** arrangements while generalizing both.





Regions and Intersections for a Pair

Let V be a real vector space, $\mathcal{A}$ an arrangement, and $\mathcal{K} \subseteq V$ an open convex set. Moreover let

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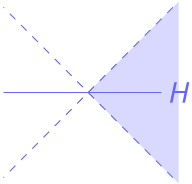
- The **regions** of the pair $(\mathcal{A}, \mathcal{K})$ are the regions of the arrangement which have nonempty intersection with $\mathcal{K}$, i.e.

$$\mathcal{R}(\mathcal{A}, \mathcal{K}) = \{R \in \mathcal{R}(\mathcal{A}) \mid R \cap \mathcal{K} \neq \emptyset\}$$



- The **intersections** of $\mathcal{C}$ are the intersections $X \in \mathcal{L}(\mathcal{A})$ which cut through $\mathcal{K}$, i.e.,

$$\mathcal{L}(\mathcal{A}, \mathcal{K}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{K} \neq \emptyset\}.$$

Zaslavsky's Theorem for Pairs

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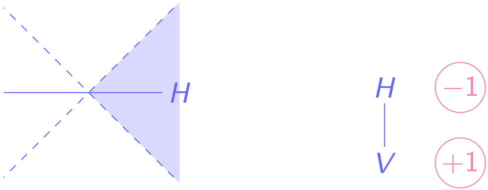

Zaslavsky's theorem says: $1 + 1(1) = 2$.

The Poincaré Polynomial of a Pair

Define the **Poincaré polynomial** of a pair $(\mathcal{A}, \mathcal{K})$ in an arrangement by

$$\mathrm{Poin}(\mathcal{A}, \mathcal{K}; t) = \sum_{X \in \mathcal{L}(\mathcal{A}, \mathcal{K})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

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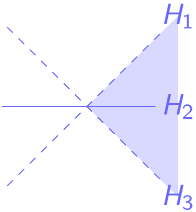
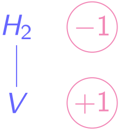
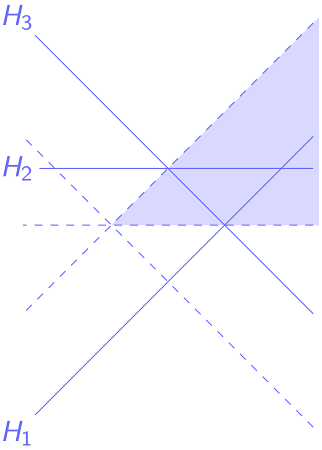


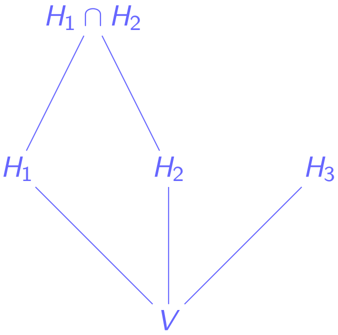


The Poincaré polynomial of this pair is $\mathrm{Poin}(\mathcal{A}, \mathcal{K}; t) = 1 + 1t$.

Example

Below is an example of a pair, together with its intersection poset





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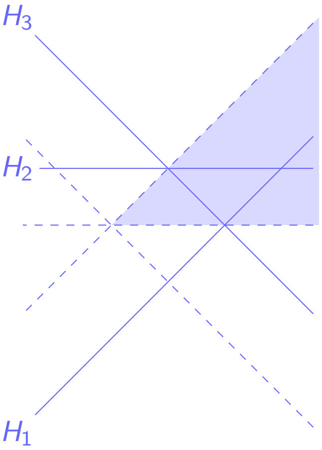

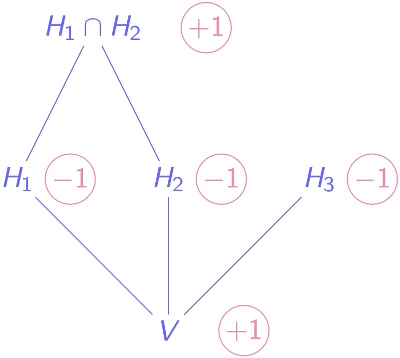

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The Varchenko-Gelfand Ring

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based on joint work with **Nick Proudfoot** and **Jayden Wang**

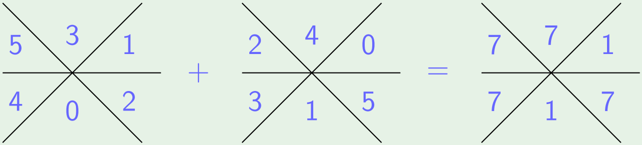
arXiv 2208.04855

A Ring from Regions

Definition

The Varchenko-Gelfand ring of $\mathcal{A}$ is the set of maps $f : \mathcal{R}(\mathcal{A}) \to \mathbb{Z}$ with pointwise addition and multiplication.

Example



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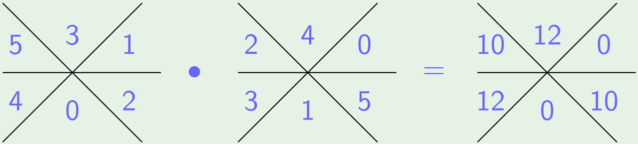

Generators for the Varchenko-Gelfand ring

Choose a set of normal vectors such that n_H is the normal vector to $H \in \mathcal{A}$. Define a *Heaviside function*

$$x_H(v) = \begin{cases} 1 & \text{if } \langle v, n_H \rangle > 0 \\ 0 & \text{else.} \end{cases}$$

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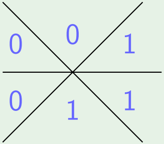
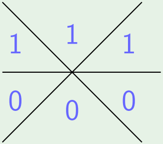
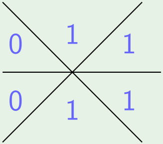
We can define this instead on regions, by choosing a representative point $v \in R$ for each region and defining $x_H(R) = x_H(v)$.

Example

$x_1 =$ 

$x_2 =$ 

$x_3 =$ 

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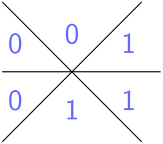
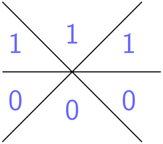
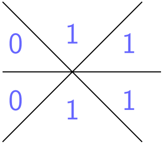
Lemma

Together with 1*, these Heaviside functions generate the Varchenko-Gelfand ring as a* $\mathbb{Z}$*-algebra.*

$x_1 =$ 

$x_2 =$ 

$x_3 =$ 

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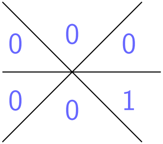

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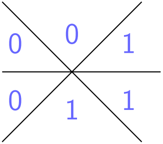

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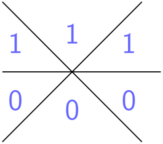

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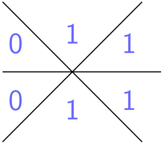

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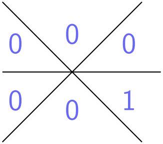

A Filtration by Degree

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$.

- We just saw that the Varchenko-Gelfand ring is generated by Heaviside functions defined by the hyperplanes of $\mathcal{A}$.
- It also has a filtration $\mathcal{F} : F_0 \subseteq F_1 \subseteq \cdots$ by degree, i.e., the collection of additive groups

$$
\begin{aligned}
F_0 &= \mathbb{Z} - \text{span}\{1\} \\
F_1 &= \mathbb{Z} - \text{span}\{1\} \cup \{x_H \mid H \in \mathcal{A}\} \\
&\vdots \\
F_i &= \mathbb{Z} - \text{span}\{\text{monomials of degree} \leq i\}.
\end{aligned}
$$

- The **associated graded ring** is $\mathcal{V}(\mathcal{A}) = \bigoplus_{i \geq 0} F_i / F_{i-1}$.

Two Classical Results

Theorem (Varchenko-Gelfand)

Each graded component F_i/F_{i-1} of $\mathcal{V}(\mathcal{A})$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis indexed by the no broken circuit sets of the arrangement.

Theorem (Rota)

For $X \in \mathcal{L}(\mathcal{A})$, we have

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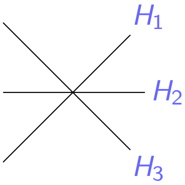
Gelfand-Rybnikov extended Varchenko-Gelfand's work to *oriented matroids*. Rota's theorem still holds in that setting, and the Hilbert series is the Poincaré polynomial of the oriented matroid.

Example

Consider the arrangement in $\mathbb{R}^2$ with normal vectors $v_1 = (1, -1)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$ (drawn below, left).



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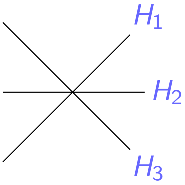

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Varchenko-Gelfand showed that

$$\mathcal{V}(\mathcal{A}) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1,\ x_2,\ x_3\} \oplus \mathbb{Z} \cdot \{x_1x_2,\ x_1x_3\}$$

where $\mathbb{Z} \cdot \{-\}$ denotes the $\mathbb{Z}$-span of $-$. Then the Hilbert series is

$$\text{Hilb}(\mathcal{V}(\mathcal{A}), t) = 1 + 3t + 2t^2$$

which matches the Poincaré polynomial we computed earlier.

Varchenko–Gelfand Ring of a Pair

Definition

The Varchenko–Gelfand ring of a pair $(\mathcal{A}, \mathcal{K})$ is the set of maps $f : \mathcal{R}(\mathcal{A}, \mathcal{K}) \to \mathbb{Z}$ with pointwise addition and multiplication.

As in the original setting, this ring is generated by Heaviside functions and admits a Heaviside filtration.

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Theorem ((DB)PW, 2022)

Let E be the set of hyperplanes that cut through $\mathcal{K}$ and $R := \mathbb{Z}[e_i \mid i \in E]$, we have isomorphisms

$$\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong R \big/ I_{(\mathcal{A},\mathcal{K})}$$

$$\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong R \big/ J_{(\mathcal{A},\mathcal{K})}$$

where the three quotienting ideals depend only on the **conditional oriented matroid** *of the pair.*

What is a conditional oriented matroid?

The short version:

- The combinatorics of a hyperplane arrangement $\mathcal{A}$ is captured by an **oriented matroid**.
- The combinatorics of a pair $(\mathcal{A}, \mathcal{K})$ is captured by a conditional oriented matroid.

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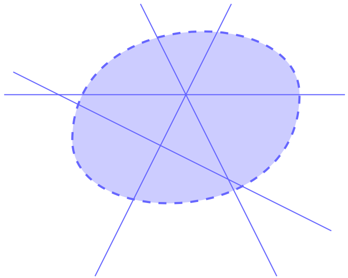

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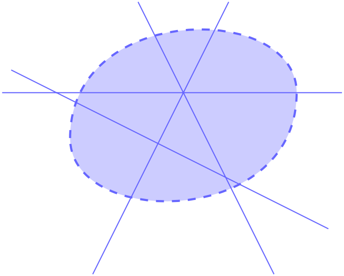

To make this precise, we need a few vocabulary items...

Signed Sets

Let E be a finite set. Recall,

- A **signed set** is an ordered pair $X = (X^+, X^-)$ of disjoint subsets.
- The **support** of $X = (X^+, X^-)$ is $\underline{X} := X^+ \cup X^-$.

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- The **composition** $X \circ Y$ of two signed sets is a signed set defined by

$$(X \circ Y)_i := \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{otherwise} \end{cases} \qquad \text{for all } i \in E.$$

where $X_i = +$ if $i \in X^+$, $X_i = -$ if $i \in X^-$ and $X_i = 0$ otherwise.

Conditional Oriented Matroids

Let E be a finite set.

Definition

A **conditional oriented matroid** on the ground set E is a collection $\mathcal{L}$ of signed sets, called **covectors**, satisfying both of the following two conditions:

- If $X, Y \in \mathcal{L}$, then $X \circ -Y \in \mathcal{L}$.
- If $X, Y \in \mathcal{L}$ and $i \in \mathrm{Sep}(X, Y)$, then there exists $Z \in \mathcal{L}$ with $Z_i = 0$ and $Z_j = (X \circ Y)_j$ for all $j \in E \setminus \mathrm{Sep}(X, Y)$.

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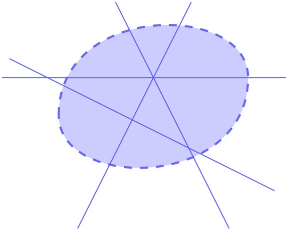

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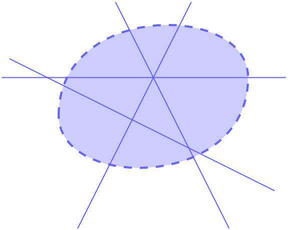

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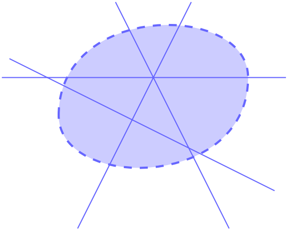

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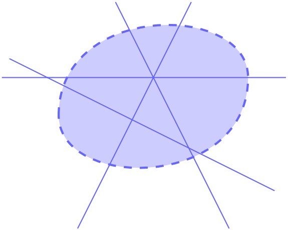

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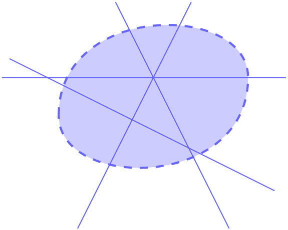

Question. What is the analogue of the Gelfand-Rybnikov ring for a COM?

Now replace chambers with **topes** which are signed sets $X \in \mathcal{L}$ whose support is the whole ground set.

Gelfand-Rybnikov Ring

Let $\mathcal{L}$ be a conditional oriented matroid.

Definition

The **Gelfand-Rybnikov** ring of $\mathcal{L}$ is the set of maps

$$f : \{\text{topes of } \mathcal{L}\} \to \mathbb{Z}$$

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Theorem ((DB)PW, 2022)

For $R := \mathbb{Z}[e_i \mid i \in E]$, we have

$$\mathrm{GR}(\mathcal{L}) \cong R \Big/ I_{\mathcal{L}} \qquad \text{and} \qquad \mathrm{gr}\,\mathrm{GR}(\mathcal{L}) \cong R \Big/ J_{\mathcal{L}}.$$

where these ideals come from the set of signed sets X such that

$$X \circ Y \notin \mathcal{L} \qquad \text{for all } Y \in \mathcal{L}.$$

Special Case: Catalan Numbers

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based on joint work with **Christian Stump**

arXiv 2204.05829

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The **Shi arrangement** of associated to Φ^+ has hyperplanes

$$H_{\beta,k} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = k\}$$

for $\beta \in \Phi^+$ and $k = 0, 1$.

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Example

The (Type A) **Shi arrangement** $\mathrm{Shi}(\Phi^+)$ has hyperplanes

$$H_{i,j,k} = \{x \in \mathbb{R}^n \mid x_i - x_j = k\}$$

for $i < j \in [n] := \{1, 2, \ldots, n\}$ and $k = 0, 1$.

Weyl Cones

Every Shi arrangement has a **reflection subarrangement** with hyperplanes

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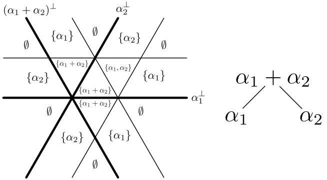
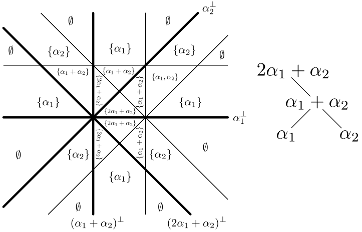
On the right, we show the Type A and Type B Shi arrangements (in rank 2). The hyperplanes of the reflection subarrangement are **bolded**.





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The Weyl cones of $\mathrm{Shi}(\Phi^+)$ are in bijection with the elements of the corresponding Weyl group W.

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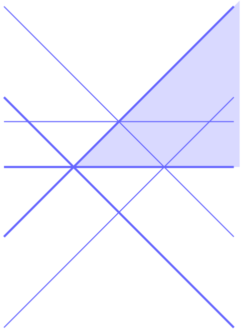
On the right, we draw the A_2 Shi arrangement, and shade the dominant cone ($=$ Weyl cone associated to $123 \in \mathfrak{S}_n$).



Regions of Weyl Cones

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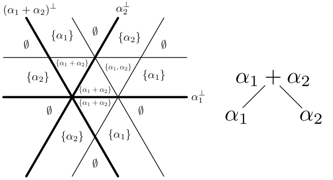
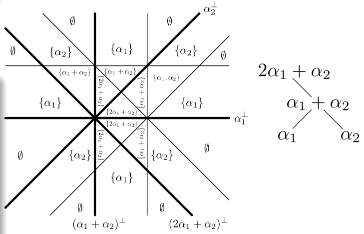


Theorem (Armstrong-Reiner-Rhoades)

For $w \in W$, the regions of the Weyl cone are in bijection with antichains of

$$\Phi^+ \backslash inv(w^{-1})$$

where $inv(w^{-1})$ is the inversion set of w^{-1}.



Intersection Posets of Weyl Cones

Theorem ((DB)S 2022)

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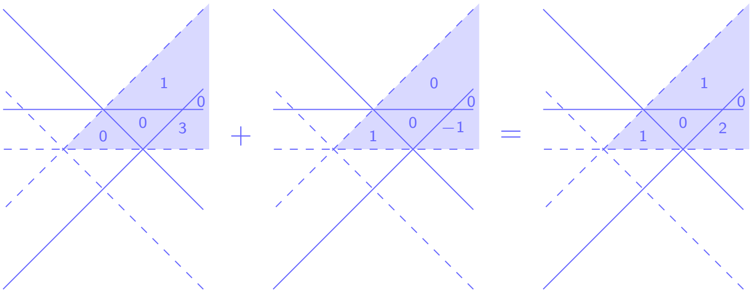
  In the remainder of this talk, I want to tell you a bit about the algebraic proof.

Back to the Varchenko–Gelfand Ring



Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton)

When C is the domiant cone of $\mathrm{Shi}(\Phi^+)$*, there exists an ideal* $I_{\Phi^+} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ *such that*

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+}$$
$$\mathfrak{gr} VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg} I_{\Phi^+})$$

In particular, both have bases indexed by antichains and

$$\mathrm{Hilb}(\mathfrak{gr} VG(wC); t) = \sum_{\substack{anitchains \\ A \subseteq \Phi^+}} t^{\#A}.$$

Once you know what to look for, Chapoton's argument has the following easy extension to all Weyl cones.

Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton + Armstrong-Reiner-Rhoades)

Let W be the Weyl group associated to Φ^+ and $w \in W$. Then there exists an ideal $I_{\Phi^+,w} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

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This extends to Shi deletions as well. But how to get to the Poincaré polynomial?

A General Presentation

Let $(\mathcal{A}, \mathcal{K})$ be a pair with regions $\mathcal{R}(\mathcal{A}, \mathcal{K})$. The following is a special case of the theorem from (DB)PW earlier.

Theorem (DB, 21)

For convex sets defined by intersections of halfspaces, one obtains a simpler set of generators $\mathcal{G} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that for any "compatible" monomial order

$$\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(\mathcal{G})$$
$$\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\mathrm{deg}}\mathcal{G})$$

In particular, the Hilbert series is

$$\mathrm{Hilb}(\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}); t) = \mathrm{Poin}((\mathcal{A}, \mathcal{K}), t).$$

The $\mathcal{K} = V$ case was first proved by Varchenko and Gelfand.

Combining these Results

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots. Let W be the Weyl group associated to Φ^+ and $w \in W$ and $w \in W$.

$$\mathrm{Poin}(wC, t) = \mathrm{Hilb}(\mathfrak{gr} VG(wC); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\# A}.$$

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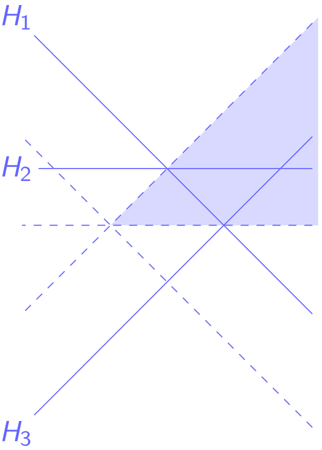
This extends to Shi deletions as well.

Let's look back at the dominant cone for Type A...

Back to Narayna Numbers I



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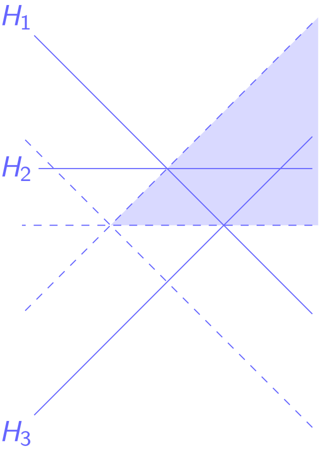

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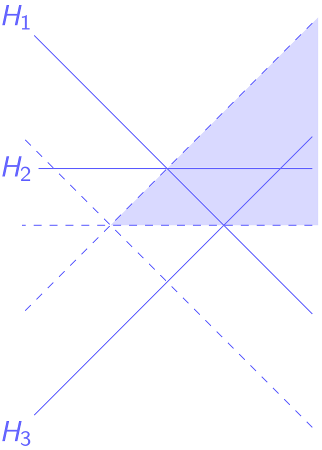

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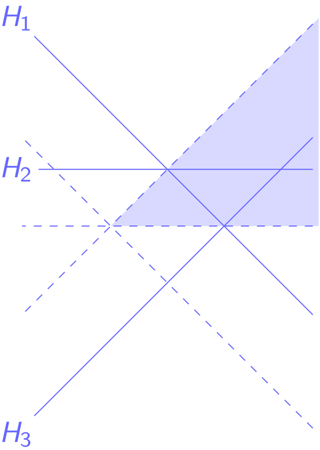
These are precisely the W-Narayana numbers.

Back to Narayna Numbers II



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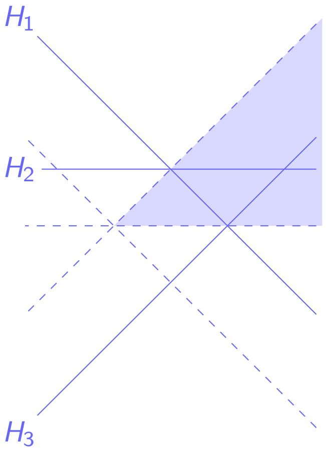

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which refine the Catalan numbers

$$C_n = \# \left\{ \text{antichains of } \Phi^+ \right\}.$$

Thank you for your attention!

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Notable Mentions

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