Cones of Hyperplane Arrangements

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contains joint work with Jang Soo Kim, Nick Proudfoot,
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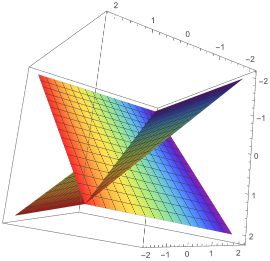
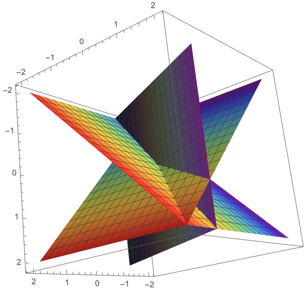
Outline

What is an arrangement of hyperplanes?

Let $V \cong \mathbb{R}^d$ be a real vector space.

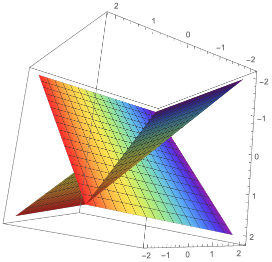
- A *hyperplane* is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.

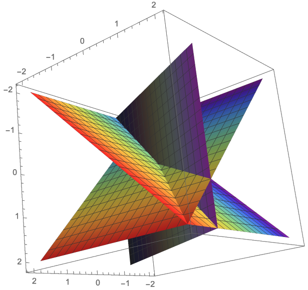




Why arrangements of hyperplanes?

- Linear optimization, linear programming
- Graph theory
- Matroid and oriented matroid theory
- Polytopes
- ...





Example (Mathematical Physics)

Recently the *residual arrangement* of a convex polytope has been useful in computing *canonical forms* associated to *positive geometries*.

(Gaetz, Kohn–Ranestad, Lam)

What kind of data could we try to compute?

$\mathcal{A}$: (central) arrangement in $\mathbb{R}^d$

$M_2 := (V \otimes \mathbb{C}) \setminus \bigcup_{H \in \mathcal{A}} (H \otimes \mathbb{C})$, and

$$\text{Poin}(\mathcal{A}, t) := \sum_{d \geq 0} \text{rank}_{\mathbb{Z}}(H^d(M_2; \mathbb{Z}))\ t^d\ .$$

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$$\mathrm{Poin}(\mathcal{A}, t) := \sum_{d \geq 0} \mathrm{rank}_{\mathbb{Z}}(H^d(M_2; \mathbb{Z}))\ t^d .$$

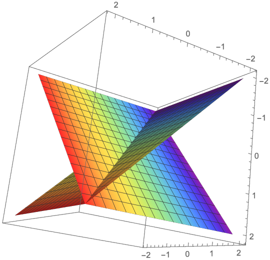
Work of Arnol'd, Brieskorn, Orlik–Solomon shows

$$\mathrm{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|\ t^{\mathrm{codim}(X)}$$

where $\mathcal{L}(\mathcal{A})$ is the set of **intersections** of $\mathcal{A}$ and $\mu(-,-)$ is the Möbius function of $\mathcal{L}(\mathcal{A})$ partially ordered by reverse inclusion.

This arrangement has 4 regions and the set of intersections is

$$\{\ \mathbb{R}^3,\ H,\ H',\ H \cap H'\ \}$$



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Work of Varchenko–Gel'fand and Moseley shows

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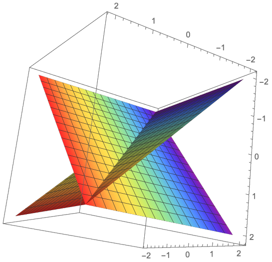

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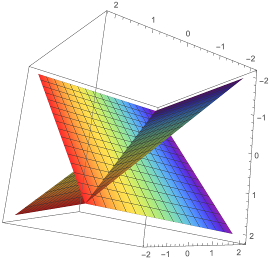
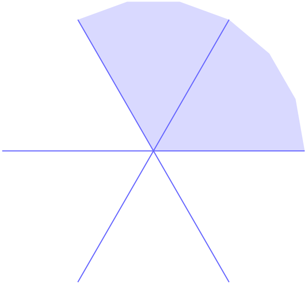
Poin$(\mathcal{A}, t)$ is related to...

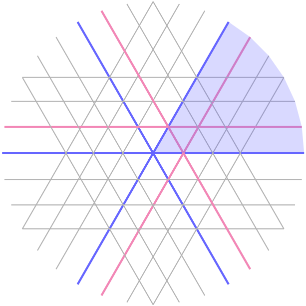
- Number of regions/bounded of the arrangement (Zaslavsky's theorem)
- Tutte polynomial,
- characteristic polynomial,
- chromatic polynomial (of a graphic arrangement),
- Orlik-Solomon algebra,
- ...



Cones of Hyperplane Arrangements

- A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of (open) halfspaces defined by some of the hyperplanes of $\mathcal{A}$.
- Cones unify the theory of **central** and **affine** arrangements while generalizing both.





The second example cannot be realized as an (affine) hyperplane arrangement.

The Poincaré Polynomial of a Cone

The **Poincaré polynomial** of a cone $\mathcal{K}$ is

$$\mathrm{Poin}(\mathcal{K}, t) = \sum_{X \in \mathcal{L}(\mathcal{K})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

where $\mathcal{L}(\mathcal{K})$ is the set of intersections X of $\mathcal{A}$ such that $X \cap \mathcal{K} \neq \emptyset$.

The Poincaré Polynomial of a Cone

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where $\mathcal{L}(\mathcal{K})$ is the set of intersections X of $\mathcal{A}$ such that $X \cap \mathcal{K} \neq \emptyset$.

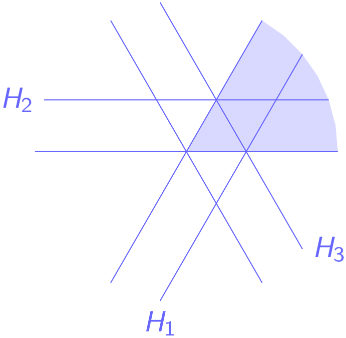
This is a reasonable thing to do:

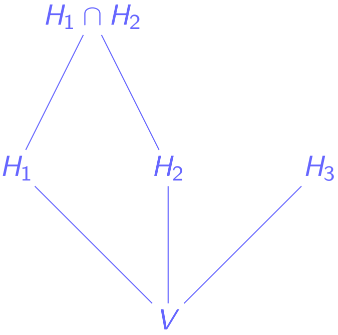
Theorem (DB–Proudfoot–Wang, '23)

There exists a topological space generalizing M_3 whose cohomology ring has Hilbert-Poincaré series is $\mathrm{Poin}(\mathcal{K}, t)$.

Example

Below (left) is an example of a cone in an affine arrangement, together with its intersection poset (right)





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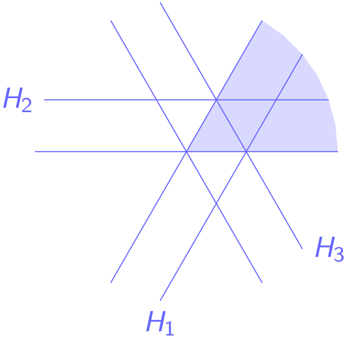
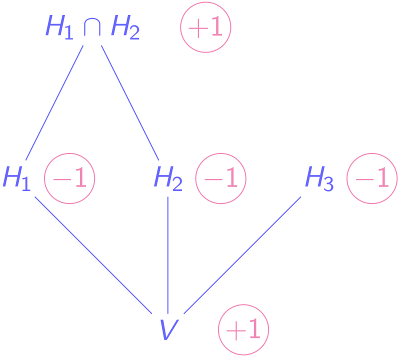
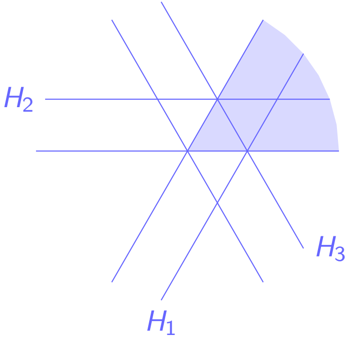




The Poincaré polynomial of this cone is $\text{Poin}(\mathcal{K}, t) = 1 + 3t + t^2$.

Example



Have: $\mathrm{Poin}(\mathcal{K}, t) = 1 + 3t + t^2$.

$$\mathrm{Poin}(\mathcal{K}, 1) = 5 = \frac{1}{3+1}\binom{2 \cdot 3}{3}$$

$=\#\{\text{increasing parking functions } (c_1, c_2, c_3)\}$

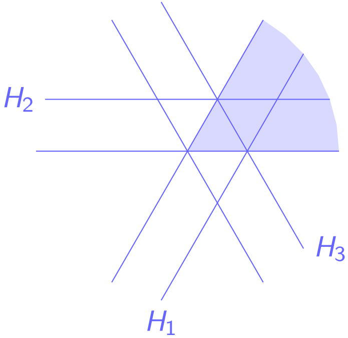
$=\#\{\text{Dyck paths of length } 6\}$

$=\#\{231\text{–avoiding permutations in } \mathfrak{S}_3\}$

$=$3rd **Catalan number**

Special case of Athanasiadis/Shi.

Example



Have: $\operatorname{Poin}(\mathcal{K}, t) = 1 + 3t + t^2$.

Moreover, the coefficients are the $n = 3$ *Narayana numbers*

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Special case of (details later):

Theorem (DB–Stump, '22)

For the Shi arrangement associated to Φ^+ $\operatorname{Poin}(eC, t)$ *has coefficients: the Narayana numbers.*

Big Picture

1. **Cones are more general than hyperplane arrangements**. The previous example is a pair which cannot be realized as an (affine) hyperplane arrangement.

2. **Cones arise naturally in mathematics (and beyond)**. We'll look at two examples coming from Coxeter groups.

3. **The cone-perspective helps solve problems**. We'll illustrate this with a numerical problem concerning the Varchenko-Gel'fand ring.

The interaction between convexity and hyperplane arrangements is well-studied...

1. **Cones of Hyperplane Arrangements**
   - Cones and Lunes (Aguiar–Mahajan),
   - Convex collections of regions (Bidigare–Hanlon–Rockmore, Brown),
   - Poset cones in the Braid arrangement (Postnikov–Reiner–Williams, Ardila-Sanchez)
   - Weyl cones for Shi and Ish arrangements (Athanasiadis, Shi, Armstrong–Reiner–Rhoades),
   - ...

2. **(Matroid-like) Axiomatizations of convexity**:
   - antimatroids/convex geometries (Edelman, Jamison),
   - convexity for oriented matroids (Las Vergnas, Sturmfels),
   - conditional oriented matroids (Bandelt-Chepoi-Knauer),
   - lopsided sets (Lawrence)
   - ...

Cones in Arrangements with Symmetries

Contains work from

- arXiv 2104.02740 (*Order*, 2021)
  joint with **Jang Soo Kim** and **Victor Reiner**
- arXiv 2208.04855 (2022)
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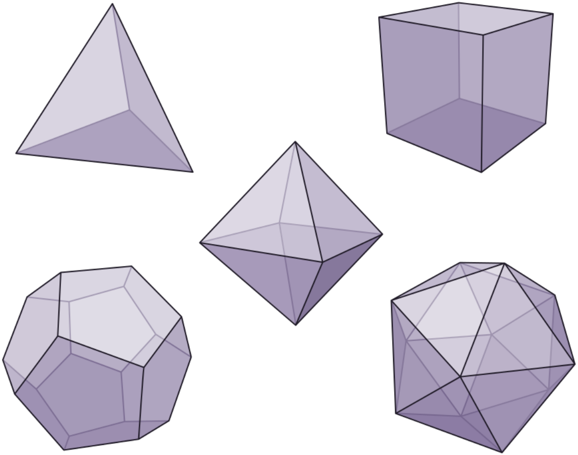
Theme:

- Poincaré polynomial known for $\mathcal{K} = V$.
- Chamber counts known for all/some cones.
- Question: what are the Poincaré polynomials of these cones?

Arrangements with Symmetries

Hyperplane arrangements and symmetry go hand in hand.



Source: Wikipedia

Arrangements with Symmetries (Example A)

Our first example comes from the symmetries of a regular n-simplex.

Definition

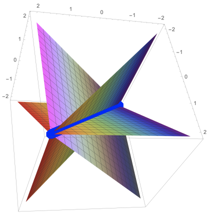
The **braid arrangement** $\mathcal{B}_n$ is the arrangement with hyperplanes

$$H_{ij} = \{x \in \mathbb{R}^n \mid x_i = x_j\}$$

for all $1 \leq i < j \leq n$.

$\mathcal{B}_3$ has hyperplanes

$$H_{13} = \{x \in \mathbb{R}^n \mid x_1 = x_3\}$$
$$H_{23} = \{x \in \mathbb{R}^n \mid x_2 = x_3\}$$
$$H_{12} = \{x \in \mathbb{R}^n \mid x_1 = x_2\}$$



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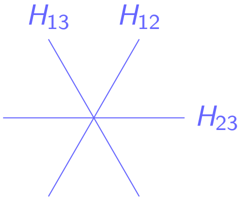

Arrangements with Symmetries (Example A)

- When $\mathcal{K} = V$:

$$\mathrm{Poin}(V, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{n - \#\mathrm{LRmax}(\sigma)}$$

where $\#\mathrm{LRmax}(\sigma)$ is the number of left-to-right maxima of $\sigma \in \mathfrak{S}_n$.

- Cones of $\mathcal{B}_n$ can be described by *partially-ordered sets* (posets) on $\{1, \ldots, n\}$, i.e., cones are

$$\mathcal{K}_P = \{x \in \mathbb{R}^n \mid x_i < x_j \Leftrightarrow i <_P j\}.$$

See: "Cone–(pre)poset dictionary" of Postnikov–Reiner–Williams.

Arrangements with Symmetries (Example A)

Question. What is the Poincaré polynomial of a poset cone $\mathcal{K}_P$?

A good answer should (a) genearalize the Poincaré polynomial of the full arrangement, and (b) depend only on P, the poset defining $\mathcal{K}_P$.

Arrangements with Symmetries (Example A)

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Theorem (DB–Kim–Reiner, '21)

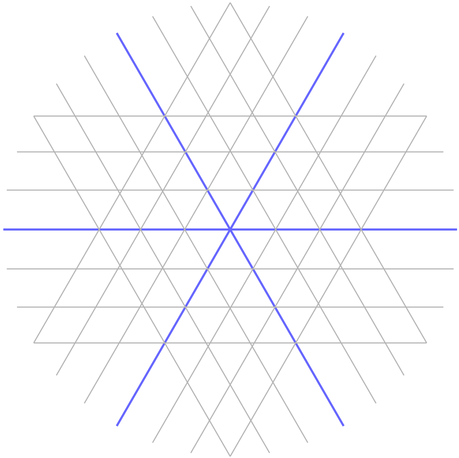
For an arbitrary poset P, there is a map $\psi : \mathcal{R}(\mathcal{K}_P) \to \mathfrak{S}_P^{\pitchfork}$ such that

$$\mathrm{Poin}(\mathcal{K}_P, t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{n - \#\mathrm{LRmax}(\psi(\sigma))}.$$

(We give this map explicitly in the paper.)

Arrangements with Symmetries (Example B)

Our second example comes from affine Weyl groups & Lie theory.



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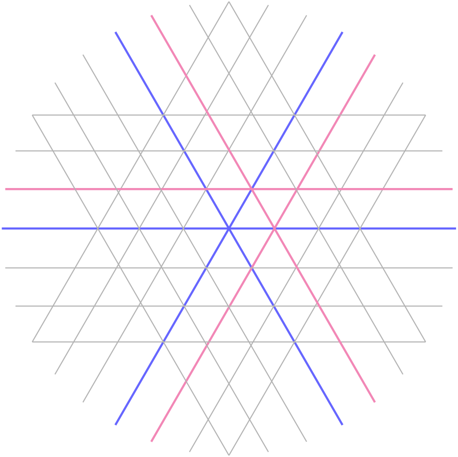

Arrangements with Symmetries (Example B)

This example comes from affine Weyl groups, Lie theory.

Definition

The nth (Type A) **Shi arrangement** Shi_n has hyperplanes

$$H_{ij} = \{x_i = x_j\} \qquad \text{and} \qquad H'_{ij} = \{x_i = x_j + 1\}$$

for $1 \leq i < j \leq n$.

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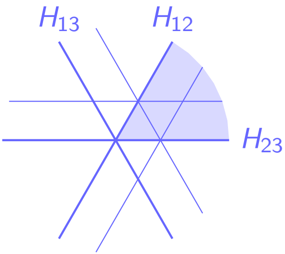
A **Weyl cone** of the Shi arrangement, is a cone which arises as a region of the **reflection subarrangement**

$$\{H_{ij} \mid 1 \leq i < j \leq n\}.$$

One Weyl cone is shaded on the right.



Arrangements with Symmetries (Example B)

Observation

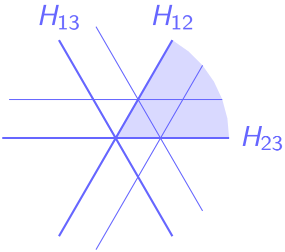
Weyl cones are in bijection with elements of the symmetric group $\mathfrak{S}_n$.

Write: $\sigma\mathcal{K}$ for the Weyl cone associated to $\sigma \in \mathfrak{S}_n$.

The $\sigma = 123$ cone is shaded on the left.

In $\mathbb{R}^3$, it is the region where $x_1 < x_2 < x_3$.

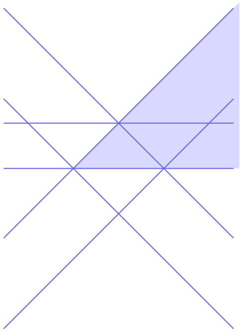
6 Weyl cones $\leftrightarrow$ $\#\mathfrak{S}_3 = 6$



Arrangements with Symmetries (Example B)

Theorem (Athanasiadis/Shi)

When $\sigma = 12\cdots n$ is the identity element of $\mathfrak{S}_n$, the number of regions in $\sigma\mathcal{K}$ is the nth Catalan number C_n.



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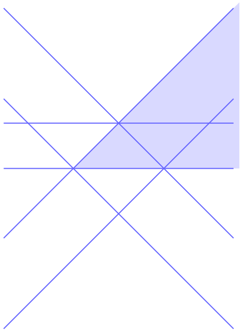

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Theorem (DB–Stump, '22)

When $\sigma = 12\cdots n$ is the identity element of $\mathfrak{S}_n$, the Whitney numbers of $\sigma\mathcal{K}$ are the Narayana numbers $N(n,k)$.

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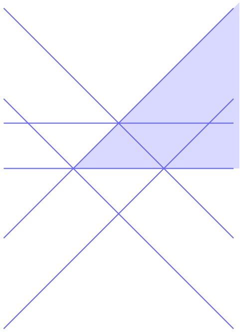

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This is a special case of one of our results, which describes all intersection posets of all Weyl cones.

Arrangements with Symmetries (Example B)

For the experts: let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots. Let W be the Weyl group associated to Φ.

Theorem (DB–Stump, '22)

For all $w \in W$, we have

$$\mathrm{Poin}(\mathrm{Shi}(\Phi^+), wC, t) = \sum_{A \in \mathrm{Anti}(\Phi^+ \setminus inv(w^{-1}))} t^{\#A}.$$

When $W = \mathfrak{S}_n$ and $w = 12 \cdots n$ (the identity permutation): Whitney numbers are the Narayana numbers.

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Some comments on the proof:

- First proof: geometry
- Second proof: commutative algebra and a general theorem about cones.

The Varchenko-Gel'fand Ring

Contains work from

- arXiv 2104.02740 (*Journal of Algebra*, 2023)
- arXiv 2208.04855 (to appear in *International Math Research Notices*) joint with **Nick Proudfoot** and **Jayden Wang**

The Varchenko-Gel'fand Ring

Plan

1. Describe $H^*(M_3(\mathcal{K}); \mathbb{Z})$ [1] algebraically using $H^0(V \setminus \bigcup_{H \in \mathcal{A}} H; \mathbb{Z})$.
2. By giving different linear bases, get different combinatorial descriptions for the coefficients of $\mathrm{Poin}(\mathcal{K}, t)$.

This allows us to give an explicit combinatorial description for the Poincaré polynomial of a Weyl cone.

[1] Thm (DB–Proudfoot–Wang, '23): This is the cohomology ring that we mentioned earlier!

The Varchenko-Gel'fand Ring of an Arrangement

The **Varchenko-Gel'fand ring** of an arrangement $\mathcal{A}$ is

$$H^0(V \setminus \bigcup_{H \in \mathcal{A}} H; \mathbb{Z}).$$

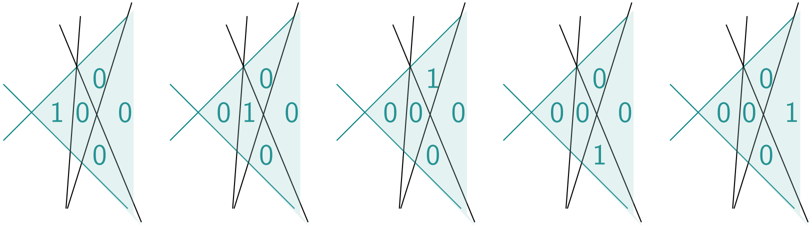
Definition

The **Varchenko-Gel'fand ring** of cone - same thing, but we only consider connected components that intersect $\mathcal{K}$.

Collection of maps $VG(\mathcal{K}) = \{f : \mathcal{R}(\mathcal{K}) \to \mathbb{Z}\}$ under pointwise addition and multiplication.

The Varchenko-Gel'fand Ring of a Cone

For every cone $\mathcal{K}$, $VG(\mathcal{K})$ has a $\mathbb{Z}$-basis of indicator functions of chambers in $\mathcal{R}(\mathcal{K})$, as shown in the example below.



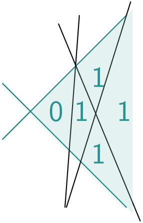
The Varchenko-Gel'fand Ring of a Cone

Pick an orientation of $\mathcal{A}$. The Varchenko-Gel'fand ring $VG(\mathcal{K})$ of a cone $\mathcal{K}$ is generated (as a $\mathbb{Z}$-algebra) by Heaviside functions

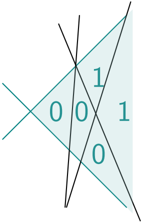
$$x_H(C) = \begin{cases} 1 & \text{if } v \in H^+ \cap \mathcal{K} \\ 0 & \text{else} \end{cases} \qquad \text{for } C \in \mathcal{R}(\mathcal{K})$$

for each hyperplane $H \in \mathcal{L}(\mathcal{K})$.

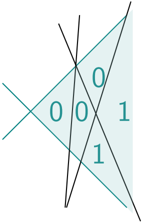
$x_1 =$



$x_2 =$



$x_3 =$



Define a map $\varphi : \mathbb{Z}[e_H \mid H \in \mathcal{A}] \to VG(\mathcal{K})$ via $e_H \mapsto x_H$.

By the previous observation, this map is surjective.

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Theorem (DB, '23)

$I_{\mathcal{K}} = \ker \varphi$ *has a simple description in terms of the oriented matroid of* $\mathcal{A}$.

- $\mathcal{K} = V$ case was proved by Varchenko and Gel'fand.
- DB–Proudfoot–Wang, '23: extend the presentation to arbitrary convex sets, define $M_3(\mathcal{K})$, *conditional oriented matroids* ($\Rightarrow$ partial answer to a problem of Bandelt-Chepoi-Knauer).

Presenting the Varchenko-Gel'fand Ring

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central arrangement and $\mathcal{K} = \bigcap_{i \in W} H_i^+$ for some $W \subseteq [n] := \{1, \ldots, n\}$.

Theorem (DB, '23)

For any graded monomial ordering on $\mathbb{Z}[e_1, \ldots, e_n]$, $I_\mathcal{K}$ has (Gröbner basis)-like generating set:*

1. $e_i^2 - e_i$ for $i \in [n]$,
2. $e_i - 1$ for $i \in [n]$ such that $i \in W$
3. Let $C = C^+ \cup C^-$ be a signed circuit.
   - If $W \cap C^\pm \neq \varnothing$ but $W \cap C^\mp = \varnothing$, then

$$\prod_{i \in C^+ \setminus W} e_i \prod_{j \in C^-} (e_j - 1) = \prod_{i \in C \setminus W} e_i - \pm l.o.t.$$

   - If $W \cap C = \varnothing$, then

$$\prod_{i \in C^+} e_i \prod_{j \in C^-} (e_j - 1) - \prod_{i \in C^+} (e_i - 1) \prod_{j \in C^-} e_j = \sum_{j \in C} \pm \prod_{i \in C - \{j\}} e_i \pm l.o.t.$$

* "Gröbner-like" means: leading term of any $f \in I_\mathcal{K}$ is divisible by the leading term of some polynomial in the generating set.

The Hilbert Series of $\mathrm{gr}_{\mathcal{F}}(VG(\mathcal{K}))$

Upshot: If $\mathcal{G}$ is the set of generators from the big theorem and $\mathcal{F}$ is the filtration by degree, then

$$\mathrm{gr}_{\mathcal{F}}(VG(\mathcal{K})) \cong \mathbb{Z}[e_1, \ldots, e_n]/\langle in_{deg}(\mathcal{G})\rangle \, .$$

Theorem (DB, '23)

The Hilbert series of $\mathrm{gr}_{\mathcal{F}}(VG(\mathcal{K}))$ *is* $\mathrm{Poin}(\mathcal{K}, t)$.

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Theorem (DB–Proudfoot–Wang, '23)

Define $M_3(\mathcal{K})$ *and prove that* $\mathrm{gr}_{\mathcal{F}}(VG(\mathcal{K})) \cong H^*(M_3(\mathcal{K}); \mathbb{Z})$.

$\mathcal{K} = V$: proved by Moseley for coefficients in $\mathbb{Q}$.

The Varchenko–Gel'fand Ring in Action

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There is an antichain basis for the (associated graded of the) Varchenko-Gel'fand ring of the Weyl cone wC.

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Combining these gives:

Theorem (DB–Stump, '22)

$$\mathrm{Poin}(wC, t) = \sum_{\substack{\text{antichains} \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\#A}$$

When w is the identity: implies that $\mathrm{Poin}(wC, t)$ is the generating function for Narayana numbers.

Future Questions

- **Scattering Amplitudes**. Canonical form of a positive geometry associated to a convex polytope can be computed from the intersections that do NOT intersect a certain closed cone.
- **Lattice Point Enumeration**. When $\mathcal{A}$ is the d-dimensional grid-arrangement, lattice points in $\mathbb{Z}^d$ are the 0-dimensional intersections.
- **Real Rootedness**. Connected to *freeness*, *supersolvability*, *Koszulity* for the (associated graded of the) Varchenko-Gel'fand ring.
- **Extended ab-index**. DB–Maglione–Stump used techniques inspired by *shellings of order complexes* to solve a conjecture motivated by *Igusa zeta functions*. We proved that a certain polynomial depending only on the intersection poset of an arrangement has nonnegative coefficients.

Thank you for your attention!