The Poincaré-extended **ab**-index

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joint with Joshua Maglione and Christian Stump
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Outline

The Big Picture

A Tale of Two Polynomials

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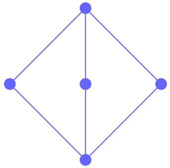

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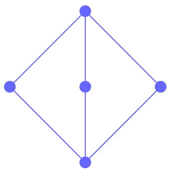

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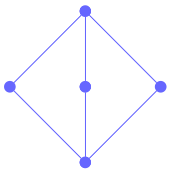

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In our example, simplifies to: $\mathrm{ex}\Psi(P;y,\mathbf{a},\mathbf{b}) = \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2$

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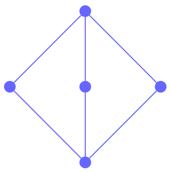

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$$\mathrm{ex}\Psi(P;0,\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} (1) \cdot \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b}) = \Psi(P;\mathbf{a},\mathbf{b})$$

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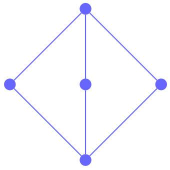

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$$\iota\left(\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})\right)|_{\mathbf{a}=1, \mathbf{b}=0} = \mathrm{Poin}(P; y)$$

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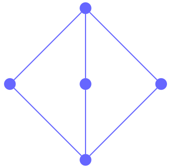

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For Coxeter arrangements (not containing an irreducible component equivalent to E_8), Maglione–Voll show that

$$\iota\left(\mathrm{ex}\Psi(\mathcal{L}(\mathcal{A}); y, \mathbf{a}, \mathbf{b})\right)|_{y=\mathbf{a}=1, \mathbf{b}=t} = \mathrm{Poin}(P; 1)\ A_{\mathrm{rank}(\mathcal{A})}(t)$$

where $A_n(t)$ denotes the nth Eulerian polynomial.

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then

$$\iota\left(\mathrm{ex}\Psi(\mathcal{L}(\mathcal{A}); y, \mathbf{a}, \mathbf{b})\right)|_{\mathbf{a}=1, \mathbf{b}=t}$$

has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! Before defining things precisely, let's look at where this comes from...

Motivation: Analytic Zeta Functions

Let $\mathcal{A}$ be a central hyperplane arrangement in a real vector space with intersection lattice $\mathcal{L}$.

Maglione–Voll prove that (after a change of variables) the **(coarse) analytic zeta function** of $\mathcal{A}$ is

$$Z_{\mathcal{A}}(y, t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{L}, \mathcal{C} \cup \{\hat{0}\}, y) \left(\frac{t}{1-t}\right)^{\#\mathcal{C}}.$$

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This is a bivariate version of the **analytic zeta function**.

A different bivariate specialization of their analytic zeta function recovers the celebrated **Motivic Zeta function** of a matroid given by Jensen–Kutler–Usatine.

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Putting all terms over the same denominator gives

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \frac{\text{Poin}(\mathcal{L}, \mathcal{C}\cup\{\hat{0}\}, y)t^{\#\mathcal{C}}(1-t)^{\text{rank}(\mathcal{A})-\#\mathcal{C}}}{(1-t)^{\text{rank}(\mathcal{A})}}.$$

The numerator of this rational function is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \text{Poin}(\mathcal{L}, \mathcal{C}\cup\{\hat{0}\}, y)t^{\#\mathcal{C}}(1-t)^{\text{rank}(\mathcal{A})-\#\mathcal{C}}.$$

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We can now state Maglione–Voll's conjecture more precisely:

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Kühne–Maglione studied $Num_{\mathcal{A}}(1,t)$ as well, and conjectured that

$$\mathrm{Poin}(\mathcal{A},1)\cdot(1+t)^{\mathrm{rank}\mathcal{A}-1} \leq Num_{\mathcal{A}}(1,t).$$

We won't discuss it today, but our proof of the Maglione-Voll conjecture will give a proof of Kühne–Maglione's conjecture (almost) for free!

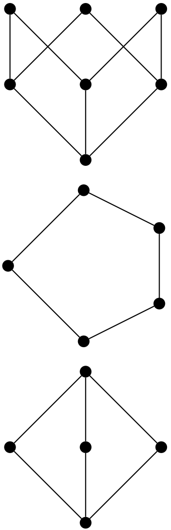
Formal Definitions

Graded Posets

Let P be a poset with $\hat{0}$ and $\hat{1}$.

- A **chain** is a subset of the ground set which is totally ordered with respect to P.
- A chain $\mathcal{C} = C_1 < C_2 < \cdots C_n$ is **maximal** if C_i covers C_{i+1} for all $i = 1, \ldots, n-1$.
- P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.
- For $x, y \in P$, the **interval** between x and y is

$$[x, y] = \{z \mid x \leq z \leq y\}.$$



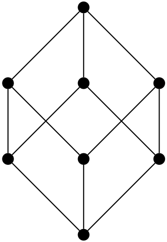
The Möbius Function of a Poset

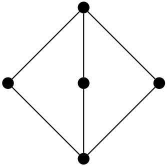
Let P be a poset with $\hat{0}$ and $\hat{1}$.

For $x \in P$, the **Möbius function** of the interval $[\hat{0}, x]$ is defined recursively by

$$\mu(\hat{0}, x) = -\sum_{\hat{0} \leq z < x} \mu(\hat{0}, z)$$

together with $\mu(\hat{0}, \hat{0}) = 1$.





The Poincaré Polynomial of a Poset

Let P be a graded poset.

Since P is graded, we can define a rank function $\mathrm{rank} : P \to \mathbb{Z}$ recursively by

$$\mathrm{rank}(\hat{0}) = 0, \text{ and}$$

$$x \lessdot z \;\Rightarrow\; \mathrm{rank}(z) = \mathrm{rank}(x) + 1$$

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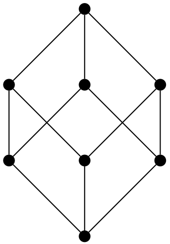
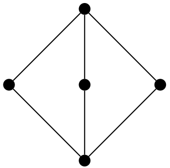
Similar to the characteristic polynomial $\chi(\mathcal{A}, t) = (-1)^{\text{rank}(\mathcal{A})} T_{\mathcal{A}}(1-t, 0)$.



$1 + 3y + 3y^2 + y^3$



$1 + 3y + 2y^2$

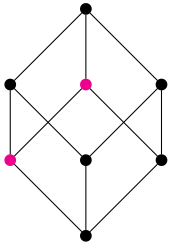
Chain Poincaré Polynomials

Let P be a graded poset and $\mathcal{C} = \{C_1 < \cdots < C_k\}$ a chain of P.

The **chain Poincaré polynomial** of $\mathcal{C}$ is

$$\mathrm{Poin}(P, \mathcal{C}; y) = \prod_{i=1}^{k} \mathrm{Poin}([C_i, C_{i+1}], y)$$

where $C_{k+1} = \hat{1}$.



$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + y)^2$$

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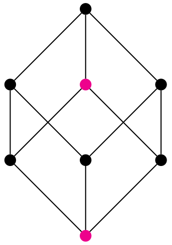

$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + 2y + y^2)(1 + y)$$

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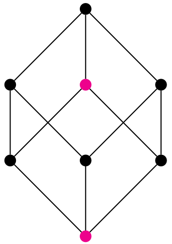

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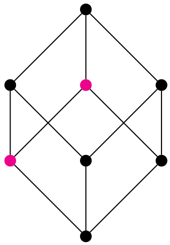
Setting $y = 1$ recovers the size of a fiber of a chain under the *support map* $z : \Sigma^*(\mathcal{A}) \to \mathcal{L}$. (Bayer-Sturmfels)

The Weight of a Chain

Let P be a graded poset and $\mathcal{C} = \{C_1 < \cdots < C_k\}$ a chain of P.

If P is rank n (every maximal chain from $\hat{0}$ to $\hat{1}$ has length $n+1$) then the **weight** of a chain $\mathcal{C}$ is $\mathrm{wt}(\mathcal{C}) = w_1 \ldots w_n \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ where

$$w_i = \begin{cases} \mathbf{b} & \text{if } \exists C_j \in \mathcal{C} \text{ such that } \mathrm{rank}(C_j) = i-1 \\ \mathbf{a} - \mathbf{b} & \text{else}. \end{cases}$$



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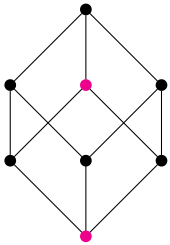

$$\mathrm{wt}(\mathcal{C}) = \mathbf{b}(\mathbf{a} - \mathbf{b})\mathbf{b}$$

The Poincaré-extended **ab**-index

Definition

Let P be a graded poset. The **(Poincaré-)extended ab-index** of P is

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

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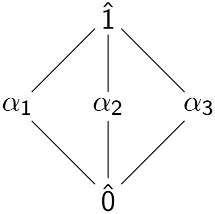

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) \\ &\quad + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2\end{aligned}$$

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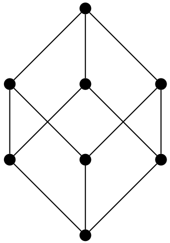
$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b}).$$



For the poset on the left:

$$\begin{aligned}\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = {} & \mathbf{a}^3 + (3y+2)\mathbf{a}^2\mathbf{b} + (3y^2+6y+2)\mathbf{aba} \\ & + (3y^2+3y+1)\mathbf{ab}^2 + (y^3+3y^2+3y)\mathbf{ba}^2 \\ & + (2y^3+6y^2+3y)\mathbf{bab} + (2y^3+3y^2)\mathbf{b}^2\mathbf{a} \\ & + y^3\mathbf{b}^3 .\end{aligned}$$

The Poincaré-extended **ab**-index

Let P be a graded poset.

Definition

The **(Poincaré-)extended ab-index** of P is

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(P; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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Their conjecture is true, even for $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! In the next part, we'll describe a family of posets where their conjecture holds.

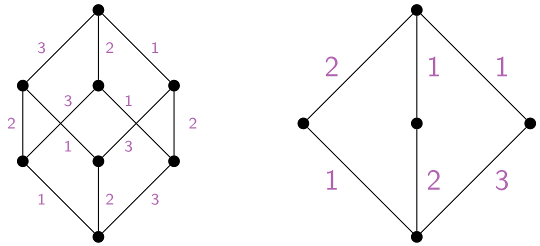
R-labeled Posets and Descent Sets

R-labelings

Let P be a graded poset, and let $\mathcal{E}(P) = \{(x, y) \mid x, y \in P,\ x \lessdot y\}$ denote the set of cover relations of P.

A labeling $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is an R-**labeling** if for every interval $[x, y]$, there is a unique maximal chain $\mathcal{M} = \{x = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = y\}$ such that the labels *weakly* increase, i.e.,

$$\lambda(C_{i-1}, C_i) \leq \lambda(C_i, C_{i+1}) \qquad \text{for } i = 2, \dots k-1.$$

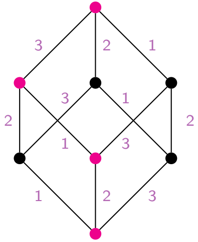


Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ.

Let $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ be a maximal chain of P. For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1}) .$$



This chain has a descent at position 1.

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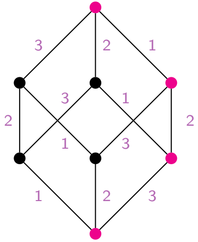


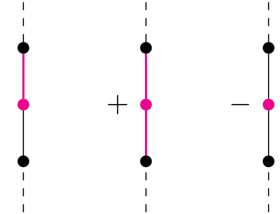
This chain has descents at positions 1 and 2.

Generalized Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ,

- $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ a maximal chain,
- E a subset of the edges of $\mathcal{M}$

For $i \in \{0, \ldots, n-1\}$, $(\mathcal{M}, E)$ has a **descent** at index i if we have one of the following situations

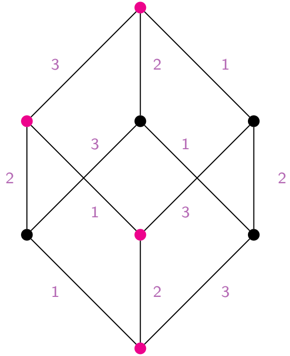


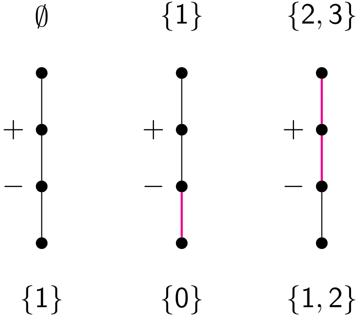
where $+$ means λ is increasing and $-$ means that λ is decreasing.

Now we include $i = 0$, which is a descent if and only if the edge above $\mathcal{M}_0$ is in E!

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





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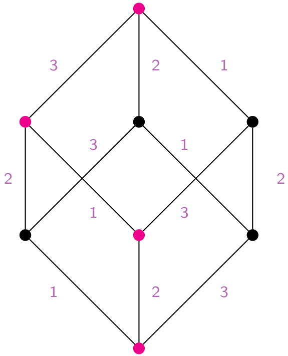
- E a subset of the edges of $\mathcal{M}$

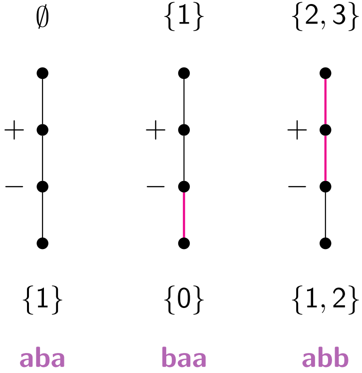
Then $\mathrm{mon}(M, E) = m_1 \dots m_n$ is the monomial in noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ with

$$m_i = \begin{cases} \mathbf{b} & \text{if } i \text{ is a descent of } (\mathcal{M}, E) \\ \mathbf{a} & \text{else}\,. \end{cases}$$

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets and monomials for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





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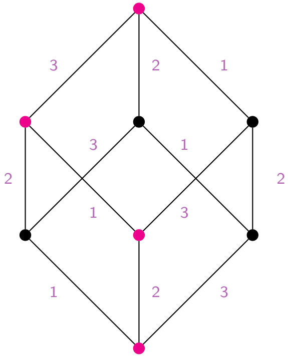
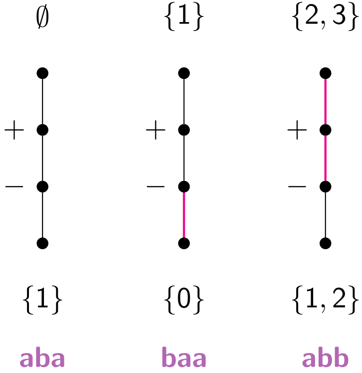




This descent statistic coincides with a statistic on *réseau* introduced by Bergeron, Mykytiuk, Sottile, and Willigenburg.

The coefficients of the extended **ab**-index

The Poincaré-extended **ab**-index

Let P be a graded poset.

Definition

The **extended ab-index** of P is

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P \setminus \{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(P; y, 1, t)$ has nonnegative coefficients.

This holds for all posets with R-labelings (theorem statement incoming).

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2023)

The extended **ab***-index of* P *is*

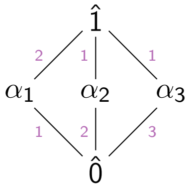
$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M},E)} y^{\#E}\mathrm{mon}(\mathcal{M}, E)$$

where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chain and* E *is a subset of its edges.*

This implies a Maglione–Voll's conjecture.

Example

Computing $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ using the theorem instead of the definition.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$$

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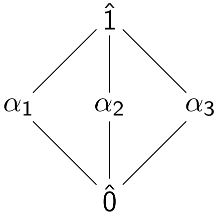
where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chains* E *is a subset of its edges.*

Connection to the **ab**-index

The (ordinary) **ab**-index

Let P be a graded poset. The **ab-index** of P is

$$\Psi(P;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } P\setminus\{\hat{1}\}} \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})\,.$$



$\mathcal{C}$	rank($\mathcal{C}$)	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})$
$\{\}$	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0}<\alpha_i\}$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\Psi(\mathcal{L};y,\mathbf{a},\mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2+\mathbf{b}(\mathbf{a}-\mathbf{b})+3\cdot(\mathbf{a}-\mathbf{b})\mathbf{b}+3\mathbf{b}^2\\ &= \mathbf{aa}+2\mathbf{ab}\end{aligned}$$

The ω-map

Definition

Let m be a monomial in **a** and **b**. Define a transformation ω that first sends **ab** to $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba}$, then all remaining **a**'s to $\mathbf{a} + y\mathbf{b}$ and all remaining **b**'s to $\mathbf{b} + y\mathbf{a}$.

If $\mathrm{m} = \mathbf{aabba}$, then

$$\omega(\mathrm{m}) = (\mathbf{a} + y\mathbf{b})(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{ba})(\mathbf{b} + y\mathbf{a})(\mathbf{a} + y\mathbf{b}).$$

By extending ω linearly, we can apply this map to sums of monomials, i.e.,

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= (\mathbf{a} + y\mathbf{b})(\mathbf{a} + y\mathbf{b}) + 2(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}) \\ &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}\,.\end{aligned}$$

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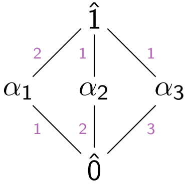
You might recognize these polynomials from earlier in this talk...

The ω-map

The **ab** index of the following poset is $\mathbf{aa} + 2\mathbf{ab}$.



We just saw that

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb} \\ &= \mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})\,.\end{aligned}$$

This is not a coincidence!

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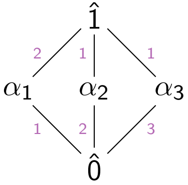

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This is not a coincidence!

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

The ω-map

Several specializations of the ω map have already appeared in the literature:

- When P is the lattice of flats of an *oriented matroid*, setting $y = 1$ recovers the ω map of Billera-Ehrenborg-Readdy,
- When P is the lattice of flats of an *oriented interval greedoid*, setting $y = 1$ recovers the ω map of Saliola-Thomas, and
- When P is a *distributive lattice*, setting $y = r - 1$ recovers the ω_r map of Ehrenborg (related to the "r-Signed Birkoff poset" from Hsiao).

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All three of these come from a pair of posets P, Q with an order- and rank- preserving surjection $z : P \to Q$ with the property that the size of the fiber $\#z^{-1}(\mathcal{C})$ of a chain $\mathcal{C}$ is an evaluation of $\mathrm{Poin}(Q, \mathcal{C}, y)$ ("versions of Bayer-Sturmfels").

Questions

- There are posets not admitting R-labelings, which have nonnegative extended **ab**-indexes. What is this larger class of posets?
- What can we say about the coefficients of anayltic zeta functions themselves (can have negative coefficients)? What about the motivic zeta functions of JKU?
- Ehrenborg–Readdy–Sloane give an analogue of the Bayer–Sturmfels relation for *arrangements on a torus*. When does the toric Poincaré-extended **ab**-index have nonnegative coefficients?
- The ω map can be reframed in terms of *peaks*. Setting $y = 1$ or $y = 0$ recovers well-studied combinatorics connected to *peak enumeration* and *quasisymmetric functions*. What can be said about y-refined peak enumerators?

Thank you for listening!

Selected References

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On the geometry of flag Hilbert-Poincaré series for matroids.
Algebraic Combinatorics (to appear), 2023.

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Flag Hilbert–Poincaré series of hyperplane arrangements and Igusa zeta functions.
Israel Journal of Mathematics (to appear), 2023.

(Overly-Simplified!) Proof Outline

Let P be a graded poset of rank n with an R-labeling λ.

Step 1: Use the following theorem to reinterpret the chain Poincaré polynomial as a sum over maximal chains with certain increasing-decreasing pattern with respect to the R-labeling.

Theorem

Let P be a poset with R-labeling λ. For $x, y \in P$ with $x < y$, we have

$$(-1)^{\mathrm{rank}(x,y)}\mu(x, y) = \#\{\textit{decreasing maximal chains in } [x, y]\}.$$

Step 2: Use inclusion-exclusion to describe the coefficients as sets.

Step 3: Show that the elements at the top of this inclusion-exclusion argument are in bijection with pairs $(\mathcal{M}, E)$.