The Varchenko-Gel'fand Ring for Weyl Cones

Galen Dorpalen-Barry

Ruhr-Universität Bochum

Arrangements in Ticino
July 1, 2022

Outline

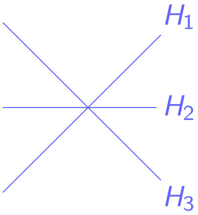
Arrangements of Hyperplanes

All vector spaces in this talk will be real!

Arrangements of Hyperplanes

All vector spaces in this talk will be real!

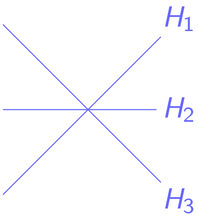
- A *hyperplane* is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.



Arrangements of Hyperplanes

All vector spaces in this talk will be real!

- A *hyperplane* is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.



Today we'll focus on

- **regions** (= open, connected components of the complement), and
- **intersections** (= nonempty intersections of some of the hyperplanes).

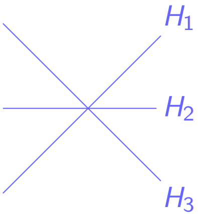
Arrangements of Hyperplanes

All vector spaces in this talk will be real!

- A *hyperplane* is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.

The following arrangement has 6 regions and the set of intersections is

$$\mathbb{R}^2, \; H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$



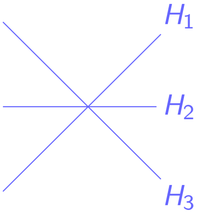
Today we'll focus on

- **regions** (= open, connected components of the complement), and
- **intersections** (= nonempty intersections of some of the hyperplanes).

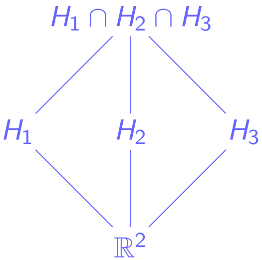
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of regions of the arrangement.



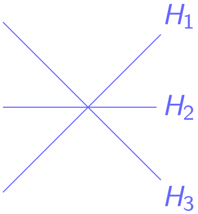
The poset of intersections is



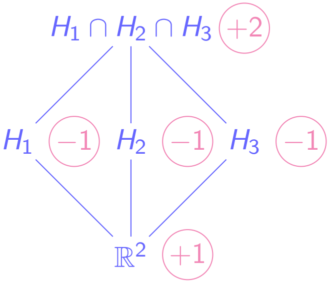
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of regions of the arrangement.



The poset of intersections is



Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

Note: Zaslavsky's theorem has two parts, depending on whether or not you include the absolute value signs.

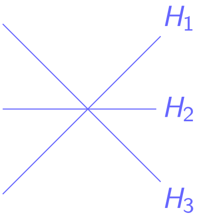
Zaslavsky's Theorem

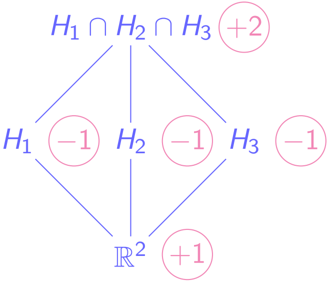
Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

Note: Zaslavsky's theorem has two parts, depending on whether or not you include the absolute value signs.





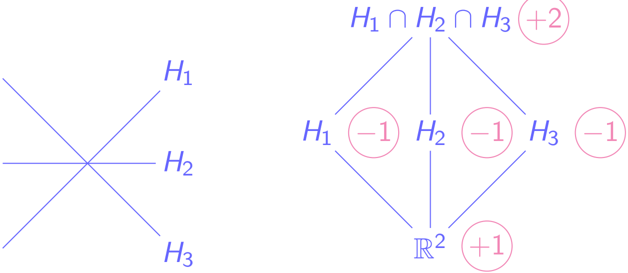
Zaslavsky's theorem says: $1 + 3(1) + 2 = 6$.

The Poincaré Polynomial

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$. Define the **Poincaré polynomial** of $\mathcal{A}$ by

$$\text{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| t^{\text{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the arrangement.

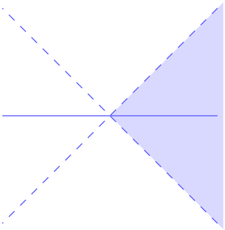


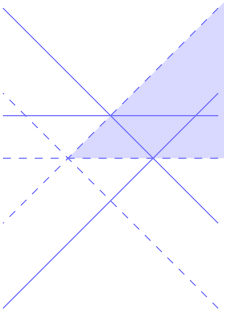
The Poincaré polynomial of this arrangement is $\text{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2$.

Cones of Hyperplane Arrangements

- A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of (open) halfspaces defined by some of the hyperplanes of $\mathcal{A}$.
- Cones are interesting in the theory of arrangements, as they unify the theory of **central** and **affine** arrangements while generalizing both.

Here are two examples of cones.





Regions and Intersections for a Cone

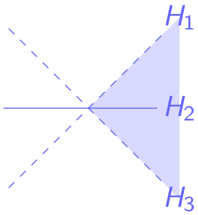
Let $\mathcal{A}$ be an arrangement in V with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$, and let $\mathcal{C}$ be a cone.

- The **regions** of $\mathcal{C}$ are the regions of the arrangement contained in $\mathcal{C}$, i.e.

$$\mathcal{R}(\mathcal{C}) = \{R \in \mathcal{R}(\mathcal{A}) \mid R \subseteq \mathcal{C}\}$$

- The **intersections** of $\mathcal{C}$ are the intersections $X \in \mathcal{L}(\mathcal{A})$ which cut through the cone, i.e.,

$$\mathcal{L}(\mathcal{C}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{C} \neq \emptyset\}.$$

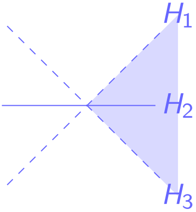


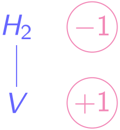
Zaslavsky's Theorem for Cones

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$, and let $\mathcal{C}$ be a cone of that arrangement.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{C}) = \sum_{X \in \mathcal{L}(\mathcal{C})} |\mu(V, X)|$$





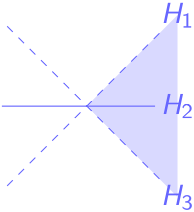
Zaslavsky's theorem says: $1 + 1(1) = 2$.

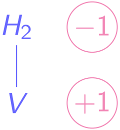
The Poincaré Polynomial of a Cone

Define the **Poincaré polynomial** of a cone $\mathcal{C}$ in an arrangement by

$$\mathrm{Poin}(\mathcal{C}, t) = \sum_{X \in \mathcal{L}(\mathcal{C})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the cone.

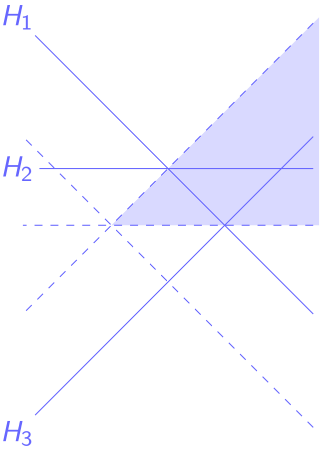


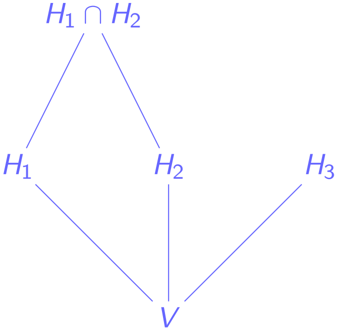


The Poincaré polynomial of this cone is $\mathrm{Poin}(\mathcal{C}, t) = 1 + 1t$.

Example: A Cone in an Affine Arrangement

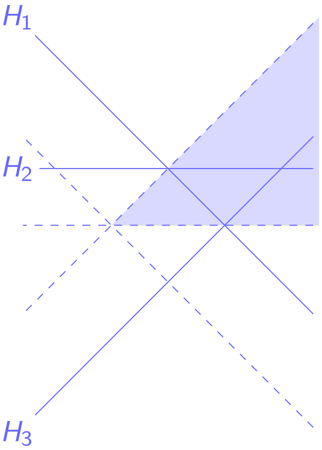
Below (left) is an example of a cone in an affine arrangement, together with its intersection poset (right)

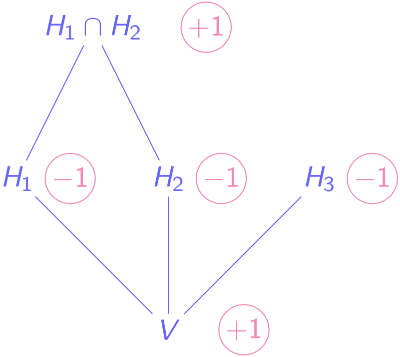




Example: A Cone in an Affine Arrangement

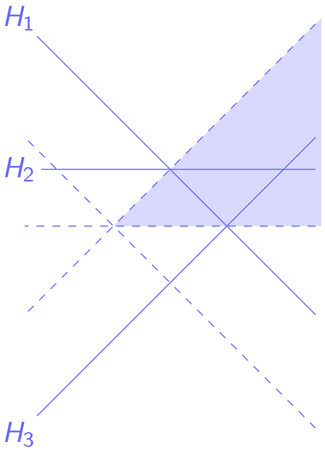
Below (left) is an example of a cone in an affine arrangement, together with its intersection poset (right)





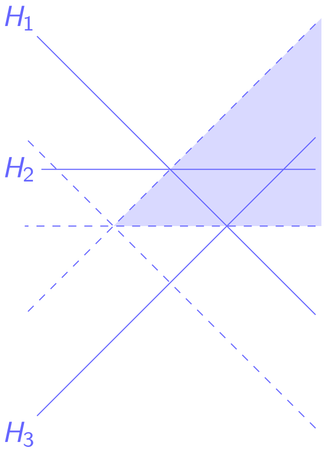
The Poincaré polynomial of this cone is $\text{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

Example Cont'd



On the previous slide, we saw that $\mathrm{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

Example Cont'd



On the previous slide, we saw that $\text{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

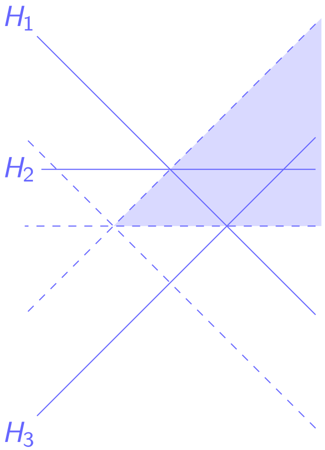
The coefficients are the $n = 3$ **Narayana numbers**

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

which refine the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Example Cont'd



On the previous slide, we saw that $\mathrm{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

The coefficients are the $n = 3$ **Narayana numbers**

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

which refine the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

This isn't a coincidence!

Shi Arrangements

Shi Arrangements

based on joint work with **Christian Stump**

arXiv 2204.05829

What is the Shi arrangement?

Let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be an irreducible crystallographic root system with a choice of positive and simple roots.

What is the Shi arrangement?

Let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be an irreducible crystallographic root system with a choice of positive and simple roots.

The **Shi arrangement** of associated to Φ^+ has hyperplanes

$$H_{\beta,k} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = k\}$$

for $\beta \in \Phi^+$ and $k = 0, 1$.

What is the Shi arrangement?

Let $\Delta \subseteq \Phi^+ \subseteq \Phi$ be an irreducible crystallographic root system with a choice of positive and simple roots.

The **Shi arrangement** of associated to Φ^+ has hyperplanes

$$H_{\beta,k} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = k\}$$

for $\beta \in \Phi^+$ and $k = 0, 1$.

Example

The (Type A) **Shi arrangement** $\mathrm{Shi}(\Phi^+)$ has hyperplanes

$$H_{i,j,k} = \{x \in \mathbb{R}^n \mid x_i - x_j = k\}$$

for $i < j \in [n] := \{1, 2, \ldots, n\}$ and $k = 0, 1$.

Weyl Cones

Every Shi arrangement has a **reflection subarrangement** with hyperplanes

$$H_{\beta,0} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}$$

for $\beta \in \Phi^+$.

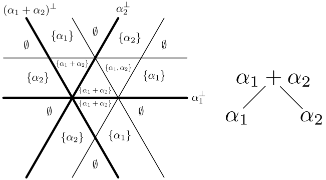
Weyl Cones

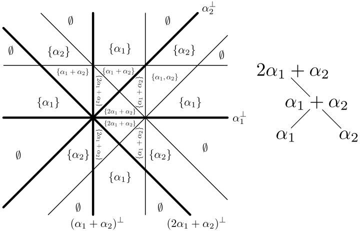
Every Shi arrangement has a **reflection subarrangement** with hyperplanes

$$H_{\beta,0} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}$$

for $\beta \in \Phi^+$.

On the right, we show the Type A and Type B Shi arrangements (in rank 2). The hyperplanes of the reflection subarrangement are **bolded**.





Weyl Cones

Every chamber of the reflection subarrangement defines a **Weyl cone** of the Shi arrangement.

Weyl Cones

Every chamber of the reflection subarrangement defines a **Weyl cone** of the Shi arrangement.

Fact

The Weyl cones of Shi(Φ^+) are in bijection with the elements of the corresponding Weyl group W.

The region associated with the identity of W is sometimes called the **dominant cone**.

Weyl Cones

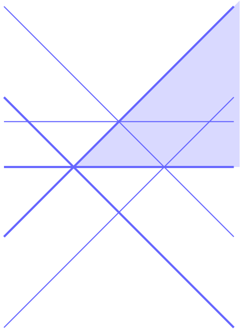
Every chamber of the reflection subarrangement defines a **Weyl cone** of the Shi arrangement.

Fact

The Weyl cones of $\mathrm{Shi}(\Phi^+)$ are in bijection with the elements of the corresponding Weyl group W.

The region associated with the identity of W is sometimes called the **dominant cone**.

On the right, we draw the A_2 Shi arrangement, and shade the dominant cone ($=$ Weyl cone associated to $123 \in \mathfrak{S}_n$).



Regions of the Dominant Cone

We view Φ^+ as a poset with order relations $\beta \prec \gamma$ if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

Regions of the Dominant Cone

We view Φ^+ as a poset with order relations $\beta \prec \gamma$ if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

Theorem (Shi (as reframed by Athanasiadis))

The regions of the dominant cone are in bijection with antichains of Φ^+.

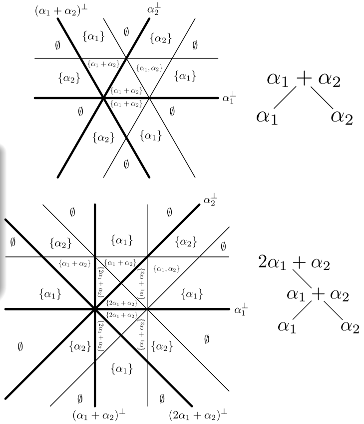
Regions of the Dominant Cone

We view Φ^+ as a poset with order relations $\beta \prec \gamma$ if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

Theorem (Shi (as reframed by Athanasiadis))

The regions of the dominant cone are in bijection with antichains of Φ^+.

On the right, we illustrate this theorem in the Type A and Type B Shi arrangements.



Regions of a Weyl Cone

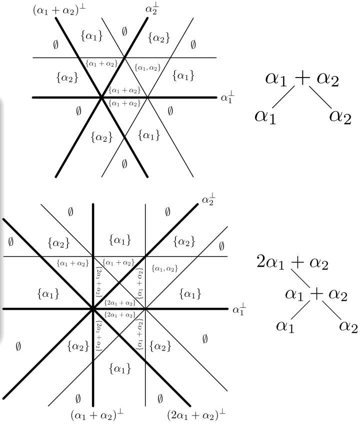
We view Φ^+ as a poset with order relations $\beta \prec \gamma$ if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

Theorem (Armstrong-Reiner-Rhoades)

For $w \in W$, the regions of the Weyl cone are in bijection with antichains of

$$\Phi^+ \backslash inv(w^{-1})$$

where $inv(w^{-1})$ is the inversion set of w^{-1}.



On the right, we illustrate this theorem in the Type A and Type B Shi arrangements.

Intersection data?

Question. What do the cone intersection posets look like?

Intersection data?

Question. What do the cone intersection posets look like?

Theorem ((DB)S 2022)

The intersection poset of wC is the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ *ordered by inclusion.*

Intersection data?

Question. What do the cone intersection posets look like?

Theorem ((DB)S 2022)

The intersection poset of wC is the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ ordered by inclusion.

Some notes on the theorem:

Intersection data?

Question. What do the cone intersection posets look like?

Theorem ((DB)S 2022)

The intersection poset of wC is the set of antichains of $\Phi^+ \setminus inv(w^{-1})$ ordered by inclusion.

Some notes on the theorem:

- The Möbus function values are ± 1 ($=$ these posets are **Eulerian**), something that is NOT true for the full intersection poset of $\mathrm{Shi}(\Phi^+)$.

Intersection data?

Question. What do the cone intersection posets look like?

Theorem ((DB)S 2022)

The intersection poset of wC is the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ *ordered by inclusion.*

Some notes on the theorem:

- The Möbus function values are ± 1 (= these posets are **Eulerian**), something that is NOT true for the full intersection poset of Shi(Φ^+).
- This result holds for all **Shi deletions** (simply delete the corresponding roots from Φ^+).

Intersection data?

Question. What do the cone intersection posets look like?

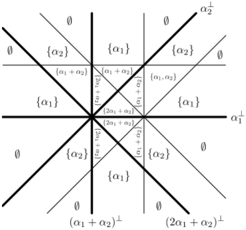
Theorem ((DB)S 2022)

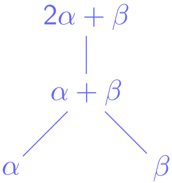
The intersection poset of wC is the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ ordered by inclusion.

Some notes on the theorem:

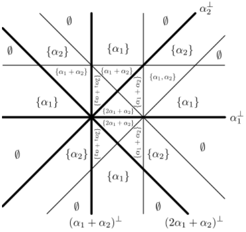
- The Möbus function values are ± 1 (= these posets are **Eulerian**), something that is NOT true for the full intersection poset of Shi(Φ^+).
- This result holds for all **Shi deletions** (simply delete the corresponding roots from Φ^+).
- The coefficient vector of the Poincaré polynomial is the f-vector of the antichain simplicial complex.

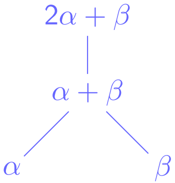
Example: B_2


$\alpha_2^\perp$
$\alpha_1^\perp$
$(\alpha_1+\alpha_2)^\perp$
$(2\alpha_1+\alpha_2)^\perp$
$\emptyset$
$\{\alpha_1\}$
$\{\alpha_2\}$
$\{\alpha_1+\alpha_2\}$
$\{\alpha_1,\alpha_2\}$
$\{2\alpha_1+\alpha_2\}$


$2\alpha+\beta$
$\alpha+\beta$
α
β

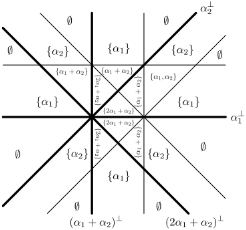
Example: B_2

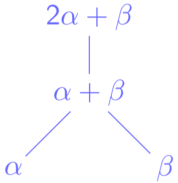




The root poset has 6 antichains
$\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha,\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

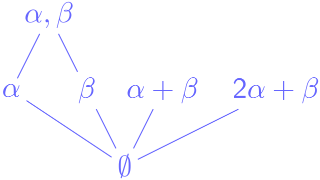
Example: B_2



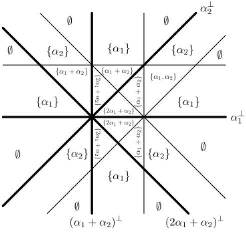


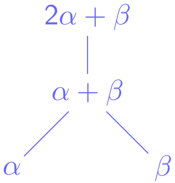
The root poset has 6 antichains
$\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha,\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

The theorem tells us that our lattice of flats of the dominant cone should look like

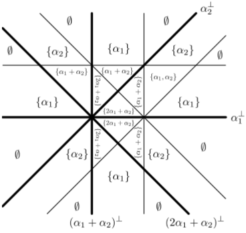


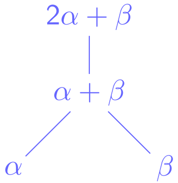
Example: B_2


$\alpha_2^\perp$
$\alpha_1^\perp$
$(\alpha_1+\alpha_2)^\perp$
$(2\alpha_1+\alpha_2)^\perp$
$\emptyset$
$\{\alpha_1\}$
$\{\alpha_2\}$
$\{\alpha_1,\alpha_2\}$
$\{\alpha_1+\alpha_2\}$
$\{2\alpha_1+\alpha_2\}$


$2\alpha+\beta$
$\alpha+\beta$
α
β

Example: B_2

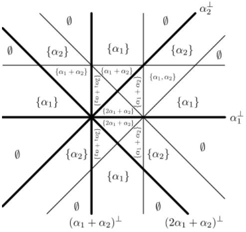


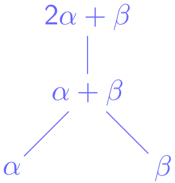


Let s_α be the reflection accross $H_{\alpha,0}$.

Then $\Phi^+ \setminus inv(s_\alpha^{-1})$ has 4 antichains $\emptyset$, $\{\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

Example: B_2

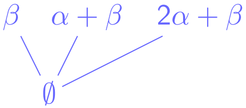




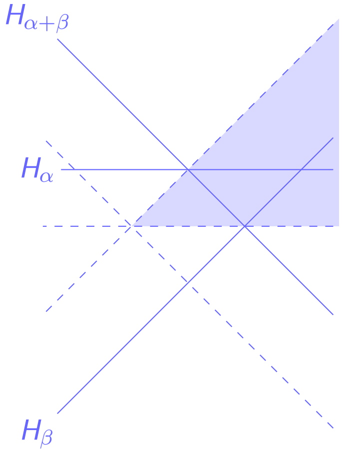
Let s_α be the reflection accross $H_{\alpha,0}$.

Then $\Phi^+ \backslash inv(s_\alpha^{-1})$ has 4 antichains $\emptyset$, $\{\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

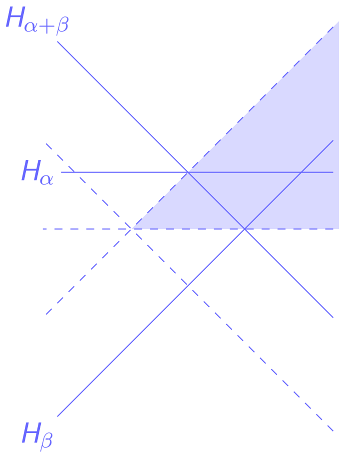
The theorem tells us that our lattice of flats of $s_\alpha C$ should look like



Back to Narayna Numbers



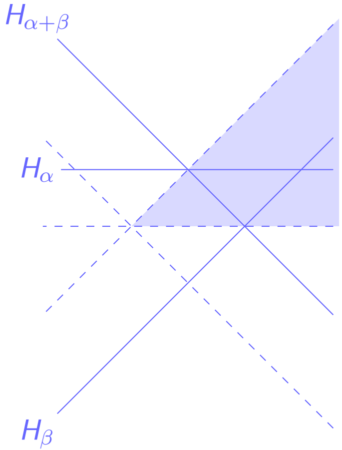
Back to Narayna Numbers



If e is the identity element of W, our theorem tells us that

$$\mathrm{Poin}(eC, t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+}} t^{\#A}.$$

Back to Narayna Numbers



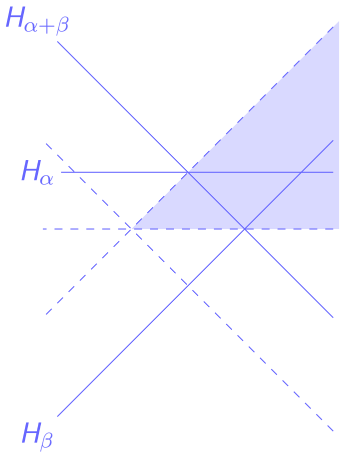
If e is the identity element of W, our theorem tells us that

$$\mathrm{Poin}(eC, t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+}} t^{\# A}.$$

Recall that the W-Narayana numbers

$$N(n,k) = \# \left\{ \begin{matrix} \text{antichains of } \Phi^+ \\ \text{of cardinality } k \end{matrix} \right\}$$

Back to Narayna Numbers



If e is the identity element of W, our theorem tells us that

$$\mathrm{Poin}(eC, t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+}} t^{\#A}.$$

Recall that the W-Narayana numbers

$$N(n, k) = \# \left\{ \begin{matrix} \text{antichains of } \Phi^+ \\ \text{of cardinality } k \end{matrix} \right\}$$

refine the W-Catalan numbers

$$C_n = \# \left\{ \text{antichains of } \Phi^+ \right\}.$$

Intersection data?

Theorem ((DB)S 2022)

The intersection poset of wC is isomorphic to the set of antichains of $\Phi^+ \setminus inv(w^{-1})$ ordered by inclusion.

Intersection data?

Theorem ((DB)S 2022)

The intersection poset of wC is isomorphic to the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ ordered by inclusion.

Some comments on the proof:

Intersection data?

Theorem ((DB)S 2022)

The intersection poset of wC is isomorphic to the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ ordered by inclusion.

Some comments on the proof:

- This theorem has an elementary/geometric proof.

Intersection data?

Theorem ((DB)S 2022)

The intersection poset of wC is isomorphic to the set of antichains of $\Phi^+ \backslash inv(w^{-1})$ ordered by inclusion.

Some comments on the proof:

- This theorem has an elementary/geometric proof.
- The interpretation of the Poincaré polynomial has a second proof via commutative algebra.

Intersection data?

Theorem ((DB)S 2022)

The intersection poset of wC is isomorphic to the set of antichains of $\Phi^{+}\backslash inv(w^{-1})$ ordered by inclusion.

Some comments on the proof:

- This theorem has an elementary/geometric proof.
- The interpretation of the Poincaré polynomial has a second proof via commutative algebra.

  In the remainder of this talk, I want to tell you a bit about the algebraic proof.

The Varchenko-Gel'fand Ring

The Varchenko-Gel'fand Ring

Warning: This section uses several terms that I haven't defined. Some useful references:

- Section 1 of "Gröbner Bases and Convex Polytopes" by Sturmfels
- Chapter 2 of "Ideals, Varieties, and Algorithms" by Cox, Little, O'Shea

The (very minor) extension to polynomial rings over $\mathbb{Z}$ is given in arXiv:2104.02740.

A Ring from Regions

Let $\mathcal{A}$ be an arrangement and $\mathcal{C}$ a cone with regions $\mathcal{R}(\mathcal{C})$.

Definition

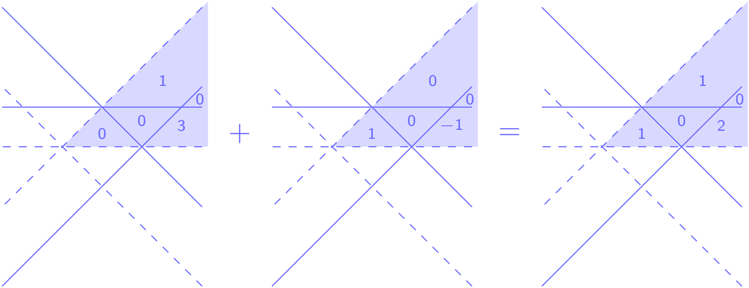
The Varchenko-Gel'fand ring of $\mathcal{C}$ is the set of maps $f : \mathcal{R}(\mathcal{C}) \to \mathbb{Z}$ with pointwise addition and multiplication.

A Ring from Regions

Let $\mathcal{A}$ be an arrangement and $\mathcal{C}$ a cone with regions $\mathcal{R}(\mathcal{C})$.

Definition

The Varchenko-Gel'fand ring of $\mathcal{C}$ is the set of maps $f : \mathcal{R}(\mathcal{C}) \to \mathbb{Z}$ with pointwise addition and multiplication.



Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton)

When C is the domiant cone of $\mathrm{Shi}(\Phi^+)$, there exists an ideal $I_{\Phi^+} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+}$$
$$\mathfrak{gr}VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg} I_{\Phi^+})$$

In particular, both have bases indexed by antichains and

$$\mathrm{Hilb}(\mathfrak{gr}VG(wC); t) = \sum_{\substack{anitchains \\ A \subseteq \Phi^+}} t^{\#A}.$$

Once you know what to look for, Chapoton's argument has the following easy extension to all Weyl cones.

Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton + Armstrong-Reiner-Rhoades)

Let W be the Weyl group associated to Φ^+ and $w \in W$. Then there exists an ideal $I_{\Phi^+,w} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+,w}$$
$$\mathfrak{gr}\, VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg} I_{\Phi^+,w})$$

In particular, both have bases indexed by antichains and

$$\mathrm{Hilb}(\mathfrak{gr}\, VG(wC); t) = \sum_{\substack{anitchains \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\#A}.$$

This extends to Shi deletions as well.

Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton + Armstrong-Reiner-Rhoades)

Let W be the Weyl group associated to Φ^+ and $w \in W$. Then there exists an ideal $I_{\Phi^+,w} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+,w}$$
$$\mathfrak{gr}\, VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg} I_{\Phi^+,w})$$

In particular, both have bases indexed by antichains and

$$\mathrm{Hilb}(\mathfrak{gr}\, VG(wC); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\#A}.$$

This extends to Shi deletions as well. But how to get to the Poincaré polynomial?

A General Presentation

Let $\mathcal{A}$ be any arrangement and $\mathcal{C}$ any cone with regions $\mathcal{R}(\mathcal{C})$.

Theorem (DB, 21)

One can explicitly describe a collection of polynomials $\mathcal{G} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that for any "compatible" monomial order

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(\mathcal{G})$$
$$\mathfrak{gr}VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg}\mathcal{G})$$

where $\mathfrak{gr}VG(\mathcal{C})$ is the associated graded of $VG(\mathcal{C})$ with respect to a certain filtration. In particular, the Hilbert series is

$$\mathrm{Hilb}(\mathfrak{gr}VG(\mathcal{C}); t) = \mathrm{Poin}(\mathcal{C}, t).$$

The $\mathcal{C} = V$ case was first proved by Varchenko and Gel'fand.

Combining these Results

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots. Let W be the Weyl group associated to Φ^+ and $w \in W$ and $w \in W$.

$$\mathrm{Poin}(wC, t) = \mathrm{Hilb}(\mathfrak{gr}VG(wC); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\#A}.$$

This extends to Shi deletions as well.

time check

Getting Off-Topic: a poset ring

A Ring from Order Ideals

Let P be a poset and $J(P)$ its collection of order ideals ($=$ down-sets).

Definition

The **order ring**[a] of P is the set of maps $f : J(P) \to \mathbb{Z}$ with pointwise addition and multiplication.

[a] I'm open to suggestions on this name!

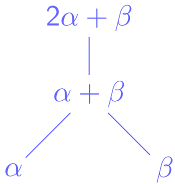
A Ring from Order Ideals

Let P be a poset and $J(P)$ its collection of order ideals (= down-sets).

Definition

The **order ring**[a] of P is the set of maps $f : J(P) \to \mathbb{Z}$ with pointwise addition and multiplication.

[a] I'm open to suggestions on this name!



This poset has 6 antichains $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha, \beta\}$, $\{\alpha + \beta\}$, and $\{2\alpha + \beta\}$.

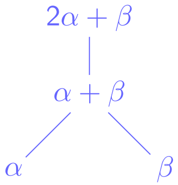
A Ring from Order Ideals

Let P be a poset and $J(P)$ its collection of order ideals ($=$ down-sets).

Definition

The **order ring**[a] of P is the set of maps $f : J(P) \to \mathbb{Z}$ with pointwise addition and multiplication.

[a] I'm open to suggestions on this name!



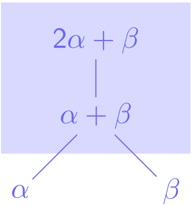
This poset has 6 antichains $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha, \beta\}$, $\{\alpha + \beta\}$, and $\{2\alpha + \beta\}$. An element of the order ring is an assignment of an integer weight to each of these order ideals.

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.



If I contains an element in the shaded region, then $x_{\alpha+\beta}(I) = 1$.

Theorem

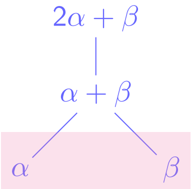
This ring is generated by Heaviside functions, i.e.

$$x_i(I) = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases}$$

for $i \in P$ and $I \in J(P)$.

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.



If I is contained in the shaded region, then $1 - x_{\alpha+\beta}(I) = 1$. The empty ideal, for example, is contained in the shaded region.

Theorem

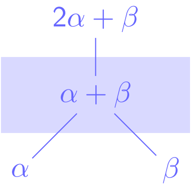
This ring is generated by Heaviside functions, i.e.

$$x_i(I) = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases}$$

for $i \in P$ and $I \in J(P)$.

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.



If I is generated by elements in the shaded region, then

$x_{\alpha+\beta}(1 - x_{2\alpha+\beta})(I) = 1$

Theorem

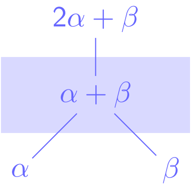
This ring is generated by Heaviside functions, i.e.

$$x_i(I) = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases}$$

for $i \in P$ and $I \in J(P)$.

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.



If I is generated by elements in the shaded region, then

$$x_{\alpha+\beta}(1-x_{2\alpha+\beta})(I)=1$$

Theorem

This ring is generated by Heaviside functions, i.e.

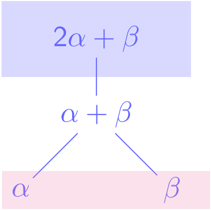
$$x_i(I)=\begin{cases}1 & \text{if } i\in I\\ 0 & \text{else}\end{cases}$$

for $i\in P$ and $I\in J(P)$. In particular, the following map is surjective

$$\mathbb{Z}[e_i \mid i\in P]\to OR(P)$$
$$e_i\mapsto x_i$$

A Ring from Order Ideals

Let $OR(P)$ be the order ring of P.



Its not hard to see why $(1-x_{\alpha+\beta})x_{2\alpha+\beta}=0$

Theorem

This ring is generated by Heaviside functions, i.e.

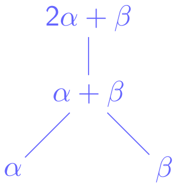
$$x_i(I)=\begin{cases}1 & \text{if } i\in I\\ 0 & \text{else}\end{cases}$$

for $i \in P$ and $I \in J(P)$. In particular, the following map is surjective

$$\mathbb{Z}[e_i \mid i\in P]\to OR(P)$$
$$e_i \mapsto x_i$$

and its kernel contains $\mathcal{G}=\{e_j\,(1-e_i)\mid i\leq_P j\}$.

A Ring from Order Ideals



This poset has 6 antichains $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha,\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

The associated graded has Hilbert series

$$1+4t+t^2.$$

Theorem

For any degree monomial order, the order ring $OR(P)$ and its associated graded (w.r.t. the degree filtration) have presentations

$$OR(P) \cong \mathbb{Z}[e_i \mid i \in P]/(\mathcal{G})$$
$$\mathfrak{gr}OR(P) \cong \mathbb{Z}[e_i \mid i \in P]/(in_{\mathrm{deg}}\mathcal{G})$$

and moreover

$$\mathrm{Hilb}(\mathfrak{gr}OR(P);t) = \sum_{\substack{A\subseteq P\\ \text{antichain}}} t^{\#A}.$$

A version of this theorem was proved by Chapoton.

The Two Rings Together

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with associated Weyl group W. Moreover, take

$w \in W$,

wC a Weyl cone, and

$E_w := \Phi^+ \backslash inv(w^{-1})$

Upshot

Combining the previous statements gives $\mathbb{Z}$-algebra isomorphisms

$$VG(wC) \cong OR(E_w)$$
$$\mathfrak{gr}VG(wC) \cong \mathfrak{gr}OR(E_w),$$

and in particular

$$\text{Poin}(wC, t) = \text{Hilb}(\mathfrak{gr}VG(wC); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq E_w}} t^{\#A}.$$

Question for the Audience

Are there other interesting arrangements/cones whose Varchenko-Gel'fand ring is isomorphic to an order ring of a poset?

Thank you for your attention!

Some References

Drew Armstrong, Victor Reiner, and Brendon Rhoades.
Parking spaces.
Adv. Math., 269:647–706, 2015.

Frédéric Chapoton.
Antichains of positive roots and Heaviside functions.
arXiv 0303220, pages 1–7, 2003.

Galen Dorpalen-Barry.
The Varchenko-Gel'fand Ring of a Cone.
arXiv 2104.02740, pages 1–16, 2021.

Jian Yi Shi.
Alcoves corresponding to an affine Weyl group.
J. London Math. Soc. (2), 35(1):42–55, 1987.

Thomas Zaslavsky.
A Combinatorial Analysis of Topological Dissections.
Advances in Math., 25(3):267–285, 1977.

Notable Mentions

Drew Armstrong and Brendon Rhoades.
The Shi arrangement and the Ish arrangement.
Trans. Amer. Math. Soc., 364(3):1509–1528, 2012.

James E. Humphreys.
Reflection Groups and Coxeter Groups, volume 29 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge, 1990.