Hyperplane Arrangements and the Varchenko-Gelfand Ring

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joint with Nick Proudfoot, Jayden Wang, and Christian Stump

UT Dallas
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Outline

1. Hyperplane Arrangements & Open, Convex Sets
2. A Ring from Regions
3. Application: Shi Arrangements

Arrangements of Hyperplanes

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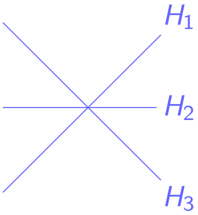
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.



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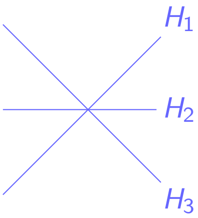

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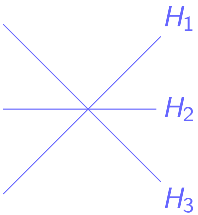
The following arrangement has 6 regions and the set of intersections is

$$\mathbb{R}^2,\ H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$



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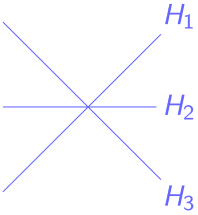
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of regions of the arrangement.



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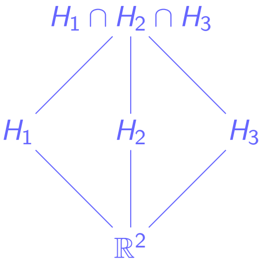

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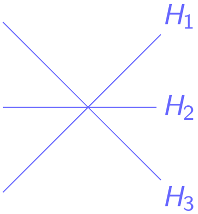

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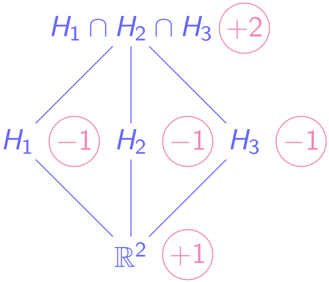

Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

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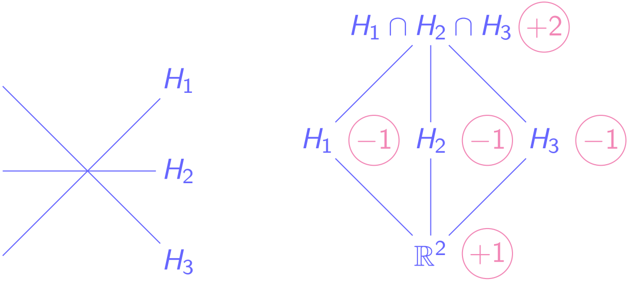


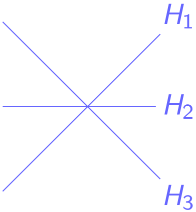
Zaslavsky's theorem says: $1 + 3(1) + 2 = 6$.

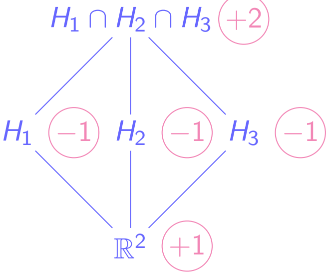
The Poincaré Polynomial

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$. Define the **Poincaré polynomial** of $\mathcal{A}$ by

$$\mathrm{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the arrangement.





The Poincaré polynomial of this arrangement is $\mathrm{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2$.

Hyperplane Arrangements and Open, Convex Sets

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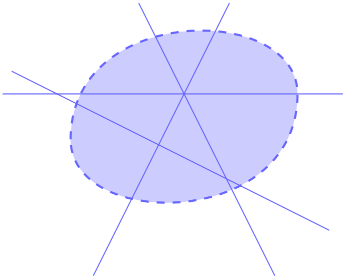
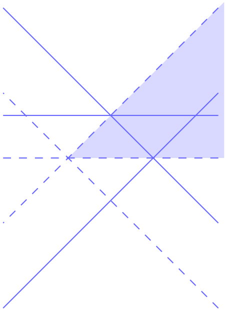
Pairs $(\mathcal{A}, \mathcal{K})$ are interesting in the theory of arrangements, as they unify the theory of **central** and **affine** arrangements while generalizing both.





Regions and Intersections for a Pair

Let V be a real vector space, $\mathcal{A}$ an arrangement, and $\mathcal{K} \subseteq V$ an open convex set. Moreover let

$\mathcal{R}(\mathcal{A})$ be the regions of $\mathcal{A}$ and

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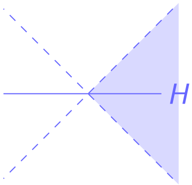
$\mathcal{L}(\mathcal{A})$ its intersections.

- The **regions** of the pair $(\mathcal{A}, \mathcal{K})$ are the regions of the arrangement which have nonempty intersection with $\mathcal{K}$, i.e.

$$\mathcal{R}(\mathcal{A}, \mathcal{K}) = \{R \in \mathcal{R}(\mathcal{A}) \mid R \cap \mathcal{K} \neq \emptyset\}$$



- The **intersections** of $\mathcal{C}$ are the intersections $X \in \mathcal{L}(\mathcal{A})$ which cut through $\mathcal{K}$, i.e.,

$$\mathcal{L}(\mathcal{A}, \mathcal{K}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{K} \neq \emptyset\}.$$

Zaslavsky's Theorem for Pairs

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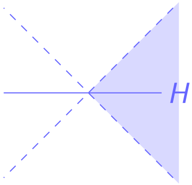
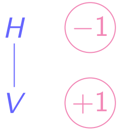




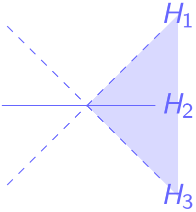
Zaslavsky's theorem says: $1 + 1(1) = 2$.

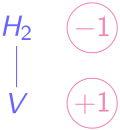
The Poincaré Polynomial of a Pair

Define the **Poincaré polynomial** of a cone $\mathcal{C}$ in an arrangement by

$$\mathrm{Poin}(\mathcal{C}, t) = \sum_{X \in \mathcal{L}(\mathcal{C})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the cone.

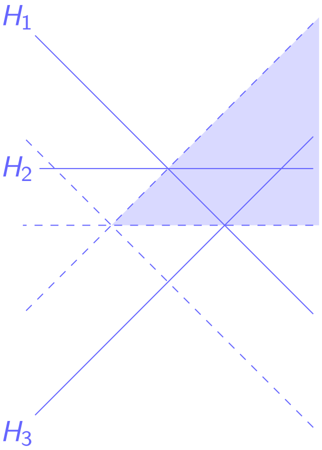


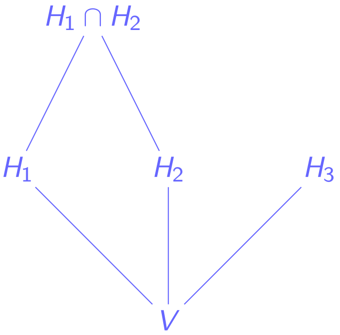


The Poincaré polynomial of this cone is $\mathrm{Poin}(\mathcal{C}, t) = 1 + 1t$.

Example

Below (left) is an example of a pair, together with its intersection poset (right)





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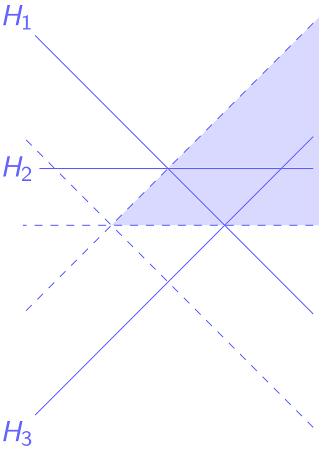

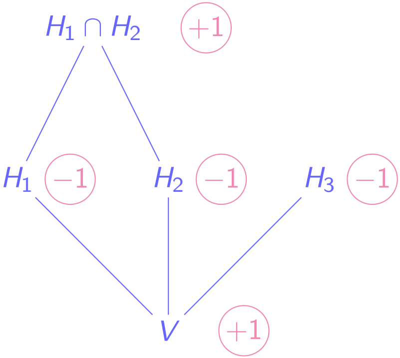

The Poincaré polynomial of this pair is $\text{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

The Varchenko-Gelfand Ring

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based on joint work with **Nick Proudfoot** and **Jayden Wang**

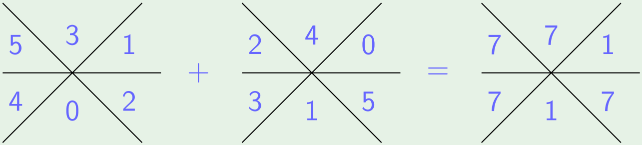
arXiv 2208.04855

A Ring from Regions

Definition

The Varchenko-Gelfand ring of $\mathcal{A}$ is the set of maps $f : \mathcal{R}(\mathcal{A}) \to \mathbb{Z}$ with pointwise addition and multiplication.

Example



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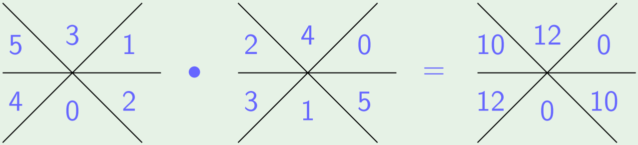

Generators for the Varchenko-Gelfand ring

Choose a set of normal vectors such that n_H is the normal vector to $H \in \mathcal{A}$. Define a *Heaviside function*

$$x_H(v) = \begin{cases} 1 & \text{if } \langle v, n_H \rangle > 0 \\ 0 & \text{else.} \end{cases}$$

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We can define this instead on regions, by choosing a representative point $v \in R$ for each region and defining $x_H(R) = x_H(v)$.

Example

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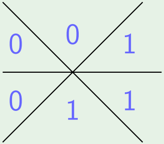

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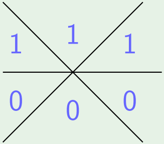

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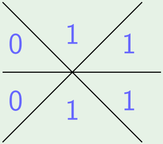

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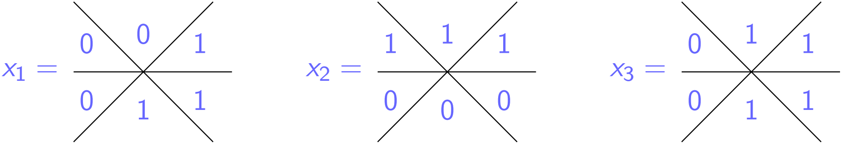
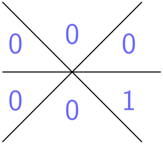
Lemma

Together with 1, these Heaviside functions generate the Varchenko-Gelfand ring as a $\mathbb{Z}$-algebra.



Let's write out the following element as a polynomial in these Heaviside functions.



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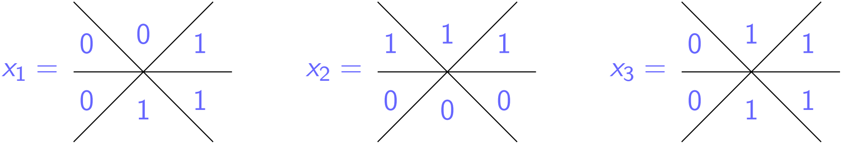

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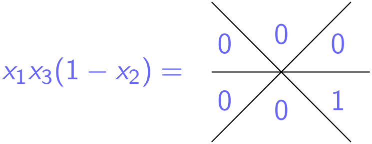

A Filtration by Degree

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$.

- We just saw that the Varchenko-Gelfand ring is generated by Heaviside functions defined by the hyperplanes of $\mathcal{A}$.
- It also has a filtration $\mathcal{F} : F_0 \subseteq F_1 \subseteq \cdots$ by degree, i.e., the collection of additive groups

$$\begin{aligned} F_0 &= \mathbb{Z} - \text{span}\{1\} \\ F_1 &= \mathbb{Z} - \text{span}\{1\} \cup \{x_H \mid H \in \mathcal{A}\} \\ &\vdots \\ F_i &= \mathbb{Z} - \text{span}\{\text{monomials of degree} \leq i\}. \end{aligned}$$

- The **associated graded ring** is $\mathcal{V}(\mathcal{A}) = \bigoplus_{i \geq 0} F_i / F_{i-1}$.

Two Classical Results

Theorem (Varchenko-Gelfand)

Each graded component F_i/F_{i-1} of $\mathcal{V}(\mathcal{A})$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis indexed by the no broken circuit sets of the arrangement.

Theorem (Rota)

For $X \in \mathcal{L}(\mathcal{A})$, we have

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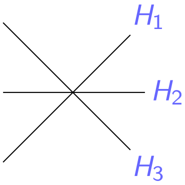
Gelfand-Rybnikov extended Varchenko-Gelfand's work to *oriented matroids*. Rota's theorem still holds in that setting, and the Hilbert series is the Poincaré polynomial of the oriented matroid.

Example

Consider the arrangement in $\mathbb{R}^2$ with normal vectors $v_1 = (1, -1)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$ (drawn below, left).



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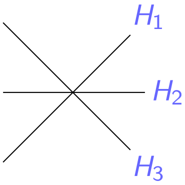

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Varchenko-Gelfand showed that

$$\mathcal{V}(\mathcal{A}) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1,\ x_2,\ x_3\} \oplus \mathbb{Z} \cdot \{x_1x_2,\ x_1x_3\}$$

where $\mathbb{Z} \cdot \{-\}$ denotes the $\mathbb{Z}$-span of $-$. Then the Hilbert series is

$$\mathrm{Hilb}(\mathcal{V}(\mathcal{A}), t) = 1 + 3t + 2t^2$$

which matches the Poincaré polynomial we computed earlier.

Varchenko–Gelfand Ring of a Pair

Definition

The Varchenko–Gelfand ring of a pair $(\mathcal{A}, \mathcal{K})$ is the set of maps $f : \mathcal{R}(\mathcal{A}, \mathcal{K}) \to \mathbb{Z}$ with pointwise addition and multiplication.

As in the original setting, this ring is generated by Heaviside functions and admits a Heaviside filtration.

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Theorem ((DB)PW, 2022)

Let E be the set of hyperplanes that cut through $\mathcal{K}$ and $R := \mathbb{Z}[e_i \mid i \in E]$, we have isomorphisms

$$\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong R \Big/ I_{(\mathcal{A},\mathcal{K})}$$

$$\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong R \Big/ J_{(\mathcal{A},\mathcal{K})}$$

where $I_{(\mathcal{A},\mathcal{K})}$ and $J_{(\mathcal{A},\mathcal{K})}$ depend only on the **conditional oriented matroid** *of the pair.*

What is a conditional oriented matroid?

The short version:

- The combinatorics of a hyperplane arrangement $\mathcal{A}$ is captured by an **oriented matroid**.
- The combinatorics of a pair $(\mathcal{A}, \mathcal{K})$ is captured by a conditional oriented matroid.

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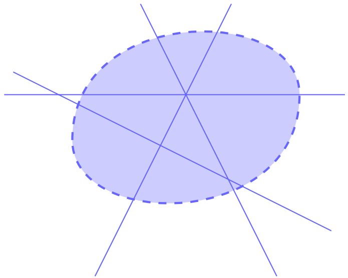

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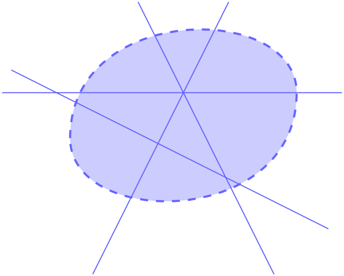

To make this precise, we need a few vocabulary items...

Signed Sets

Let E be a finite set. Recall,

- A **signed set** is an ordered pair $X = (X^+, X^-)$ of disjoint subsets.
- The **support** of $X = (X^+, X^-)$ is $\underline{X} := X^+ \cup X^-$.

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- The **support** of $X = (X^+, X^-)$ is $\underline{X} := X^+ \cup X^-$.
- The **separating set** of signed sets X, Y is the set of coordinates in the intersection of the supports at which X and Y differ, i.e.,

$$\mathrm{Sep}(X, Y) := \{i \in E \mid X_i = -Y_i \neq 0\}.$$

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- A **signed set** is an ordered pair $X = (X^+, X^-)$ of disjoint subsets.
- The **support** of $X = (X^+, X^-)$ is $\underline{X} := X^+ \cup X^-$.
- The **separating set** of signed sets X, Y is the set of coordinates in the intersection of the supports at which X and Y differ, i.e.,

$$\mathrm{Sep}(X, Y) := \{i \in E \mid X_i = -Y_i \neq 0\}.$$

- The **composition** $X \circ Y$ of two signed sets is a signed set defined by

$$(X \circ Y)_i := \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{otherwise} \end{cases} \qquad \text{for all } i \in E.$$

where $X_i = +$ if $i \in X^+$, $X_i = -$ if $i \in X^-$ and $X_i = 0$ otherwise.

Conditional Oriented Matroids

Let E be a finite set.

Definition

A **conditional oriented matroid** on the ground set E is a collection $\mathcal{L}$ of signed sets, called **covectors**, satisfying both of the following two conditions:

- If $X, Y \in \mathcal{L}$, then $X \circ -Y \in \mathcal{L}$.
- If $X, Y \in \mathcal{L}$ and $i \in \mathrm{Sep}(X, Y)$, then there exists $Z \in \mathcal{L}$ with $Z_i = 0$ and $Z_j = (X \circ Y)_j$ for all $j \in E \setminus \mathrm{Sep}(X, Y)$.

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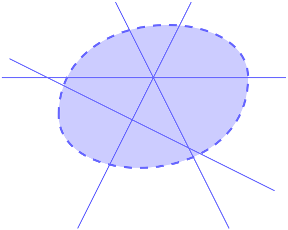

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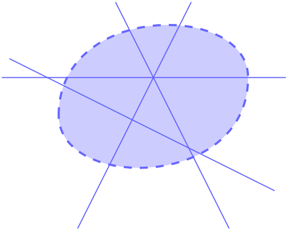

1st axiom says: $\mathcal{K}$ is open
2nd axiom says: $\mathcal{K}$ is convex

Question. What is the analogue of the Varchenko–Gelfand ring for a COM?

Gelfand-Rybnikov Ring for Conditional Oriented Matroids

A **tope** of a conditional oriented matroid $\mathcal{L}$ is a signed set $X \in \mathcal{L}$ whose support is the ground set of $\mathcal{K}$.

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The Gelfand-Rybnikov ring of conditional oriented matroid is the set of maps from the set of topes to the integers, with pointwise addition and multiplication.

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The Gelfand-Rybnikov ring of conditional oriented matroid is the set of maps from the set of topes to the integers, with pointwise addition and multiplication.

Theorem ((DB)PW, 2022)

For $R := \mathbb{Z}[e_i \mid i \in E]$, *we have canonical isomorphisms*

$$\mathrm{GR}(\mathcal{L}) \cong R \Big/ I_{\mathcal{L}}$$

$$\mathrm{gr}\,\mathrm{GR}(\mathcal{L}) \cong R \Big/ J_{\mathcal{L}}.$$

Application: Shi Arrangements

Application: Shi Arrangements

based on joint work with **Christian Stump**

arXiv 2204.05829

What is the Shi arrangement?

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Example

The (Type A) **Shi arrangement** $\mathrm{Shi}(\Phi^+)$ has hyperplanes

$$H_{i,j,k} = \{x \in \mathbb{R}^n \mid x_i - x_j = k\}$$

for $i < j \in [n] := \{1, 2, \ldots, n\}$ and $k = 0, 1$.

Weyl Cones

Every Shi arrangement has a **reflection subarrangement** with hyperplanes

$$H_{\beta,0} = \{x \in \mathbb{R}^n \mid \langle \beta, x \rangle = 0\}$$

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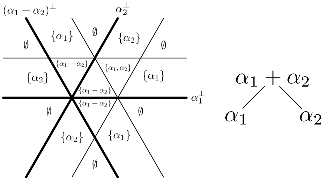
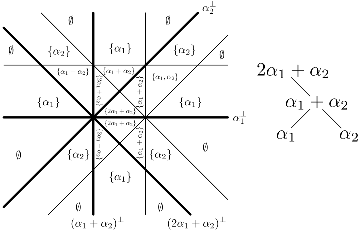
On the right, we show the Type A and Type B Shi arrangements (in rank 2). The hyperplanes of the reflection subarrangement are **bolded**.





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The Weyl cones of Shi(Φ^+) are in bijection with the elements of the corresponding Weyl group W.

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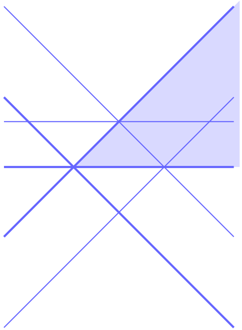
On the right, we draw the A_2 Shi arrangement, and shade the dominant cone (= Weyl cone associated to $123 \in \mathfrak{S}_n$).



Regions of Weyl Cones

The **root poset** is the poset on Φ^+ with order relations $\beta \prec \gamma$ if $\gamma - \beta$ is a nonnegative linear combination of simple roots.

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Theorem (Shi/Athanasiadis)

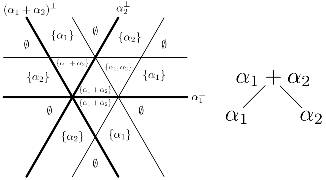
The regions of the dominant cone are in bijection with antichains of the root poset.

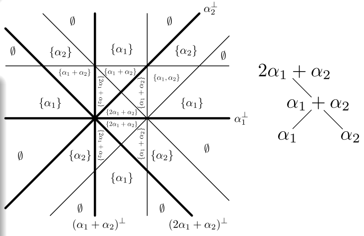
Theorem (Armstrong-Reiner-Rhoades)

For $w \in W$, the regions of the Weyl cone are in bijection with antichains of

$$\Phi^+ \backslash inv(w^{-1})$$

where $inv(w^{-1})$ is the inversion set of w^{-1}.





Intersection Posets of Weyl Cones

Theorem ((DB)S 2022)

The intersection poset of wC is the set of antichains of $\Phi^+ \setminus inv(w^{-1})$ ordered by inclusion.

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$$N(\Phi^+, k) = \# \left\{ \begin{matrix} \text{antichains of } \Phi^+ \\ \text{of cardinality } k \end{matrix} \right\}$$

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refine the W-Catalan numbers

$$C(\Phi^+) = \# \left\{ \text{antichains of } \Phi^+ \right\}.$$

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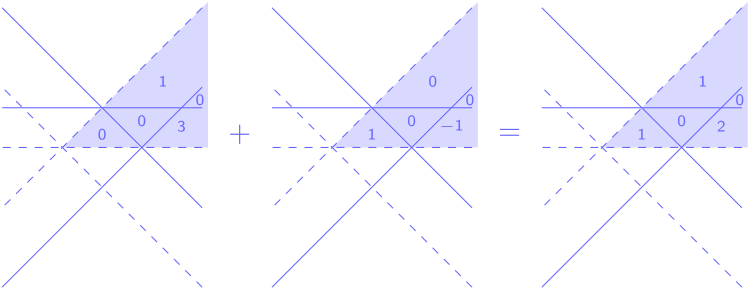
- This theorem has an elementary/geometric proof.
- The interpretation of the Poincaré polynomial has a second proof via commutative algebra.

  In the remainder of this talk, I want to tell you a bit about the algebraic proof.

Back to the Varchenko–Gelfand Ring



Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton)

When C is the domiant cone of $\mathrm{Shi}(\Phi^+)$, there exists an ideal $I_{\Phi^+} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+}$$
$$\mathfrak{gr}VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg} I_{\Phi^+})$$

In particular, both have bases indexed by antichains and

$$\mathrm{Hilb}(\mathfrak{gr}VG(wC);t) = \sum_{\substack{anitchains \\ A \subseteq \Phi^+}} t^{\#A}.$$

Once you know what to look for, Chapoton's argument has the following easy extension to all Weyl cones.

Another Presentation

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots.

Theorem (Chapoton + Armstrong-Reiner-Rhoades)

Let W be the Weyl group associated to Φ^+ and $w \in W$. Then there exists an ideal $I_{\Phi^+,w} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/I_{\Phi^+,w}$$
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This extends to Shi deletions as well.

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This extends to Shi deletions as well. But how to get to the Poincaré polynomial?

A General Presentation

Let $(\mathcal{A}, \mathcal{K})$ be a pair with regions $\mathcal{R}(\mathcal{A}, \mathcal{K})$. The following is a special case of the theorem from (DB)PW earlier.

Theorem (DB, 21)

One can explicitly describe a collection of polynomials $\mathcal{G} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that for any "compatible" monomial order

$$\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(\mathcal{G})$$
$$\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\mathrm{deg}}\mathcal{G})$$

In particular, the Hilbert series is

$$\mathrm{Hilb}(\mathrm{gr}\,\mathrm{GR}(\mathcal{A}, \mathcal{K}); t) = \mathrm{Poin}((\mathcal{A}, \mathcal{K}), t).$$

The $\mathcal{K} = V$ case was first proved by Varchenko and Gelfand.

Combining these Results

Let $\Delta \subset \Phi^+ \subset \Phi$ be an irreducible crystallographic root system with choice of simple and positive roots. Let W be the Weyl group associated to Φ^+ and $w \in W$ and $w \in W$.

$$\mathrm{Poin}(wC, t) = \mathrm{Hilb}(\mathfrak{gr} VG(wC); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq \Phi^+ \setminus inv(w^{-1})}} t^{\#A}.$$

This extends to Shi deletions as well.

Thank you for your attention!

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