Narayana numbers from the geometry of Shi arrangements

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Co(mbinatorial workspace)
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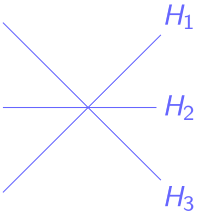
Outline

Arrangements of Hyperplanes

- A *hyperplane* is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.



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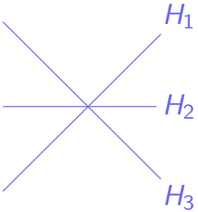

Today we'll focus on

- **regions** (= open, connected components of the complement), and
- **intersections** (= nonempty intersections of some of the hyperplanes).

Arrangements of Hyperplanes

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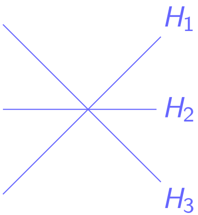
- A collection of finitely-many (distinct) hyperplanes is an *arrangement*.

The following arrangement has 6 regions and the set of intersections is

$$\mathbb{R}^2,\ H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$



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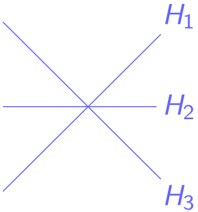
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of regions of the arrangement.



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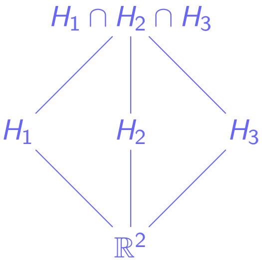

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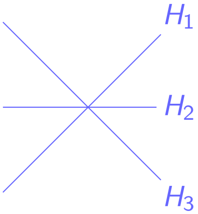

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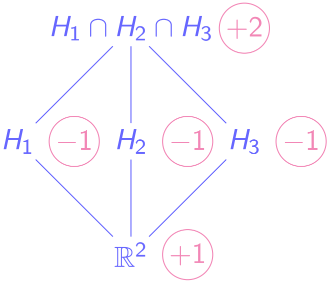

Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

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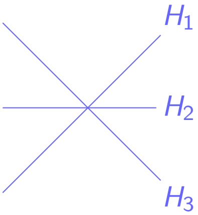
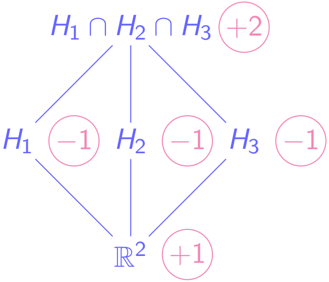




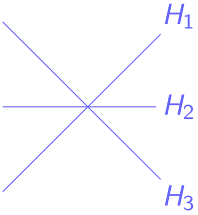
Zaslavsky's theorem says: $1 + 3(1) + 2 = 6$.

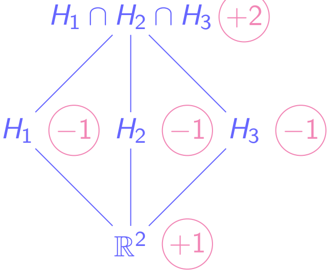
The Poincaré Polynomial

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$. Define the **Poincaré polynomial** of $\mathcal{A}$ by

$$\mathrm{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| t^{\mathrm{codim}(X)}.$$

Its coefficients are the **Whitney numbers** of the arrangement.



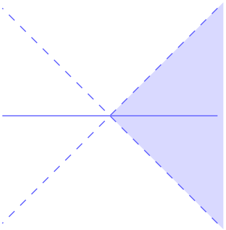


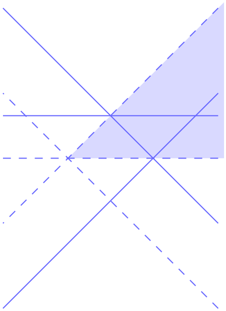
The Poincaré polynomial of this arrangement is $\mathrm{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2$.

Cones of Hyperplane Arrangements

- A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of (open) halfspaces defined by some of the hyperplanes of $\mathcal{A}$.
- Cones are interesting in the theory of arrangements, as they unify the theory of **central** and **affine** arrangements while generalizing both.

Here are two examples of cones.





Regions and Intersections for a Cone

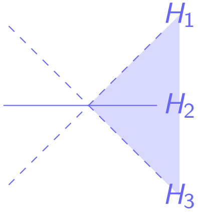
Let $\mathcal{A}$ be an arrangement in V with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$, and let $\mathcal{C}$ be a cone.

- The **regions** of $\mathcal{C}$ are the regions of the arrangement contained in $\mathcal{C}$, i.e.

$$\mathcal{R}(\mathcal{C}) = \{R \in \mathcal{R}(\mathcal{A}) \mid R \subseteq \mathcal{C}\}$$

- The **intersections** of $\mathcal{C}$ are the intersections $X \in \mathcal{L}(\mathcal{A})$ which cut through the cone, i.e.,

$$\mathcal{L}(\mathcal{C}) = \{X \in \mathcal{L}(\mathcal{A}) \mid X \cap \mathcal{C} \neq \emptyset\}.$$

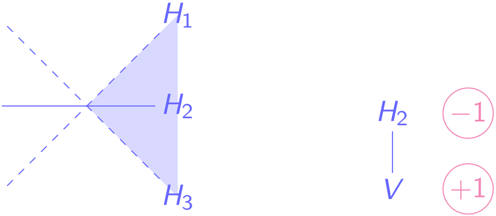


Zaslavsky's Theorem for Cones

Let $\mathcal{A}$ be an arrangement with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$, and let $\mathcal{C}$ be a cone of that arrangement.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{C}) = \sum_{X \in \mathcal{L}(\mathcal{C})} |\mu(V, X)|$$



Zaslavsky's theorem says: $1 + 1(1) = 2$.

The Poincaré Polynomial of a Cone

Define the **Poincaré polynomial** of a cone $\mathcal{C}$ in an arrangement by

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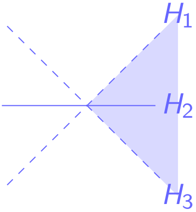
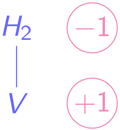
Its coefficients are the **Whitney numbers** of the cone.

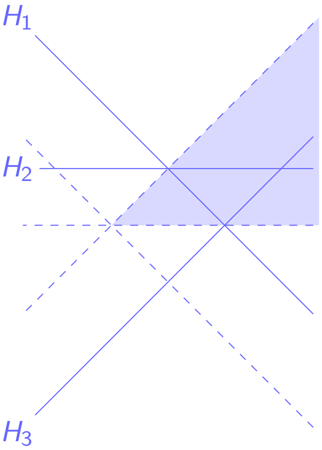


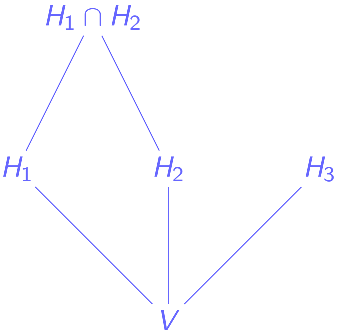


The Poincaré polynomial of this cone is $\mathrm{Poin}(\mathcal{C}, t) = 1 + 1t$.

Example: A Cone in an Affine Arrangement

Below (left) is an example of a cone in an affine arrangement, together with its intersection poset (right)





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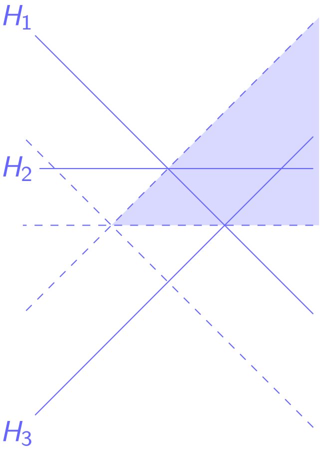
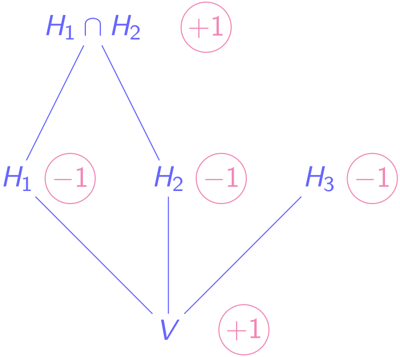
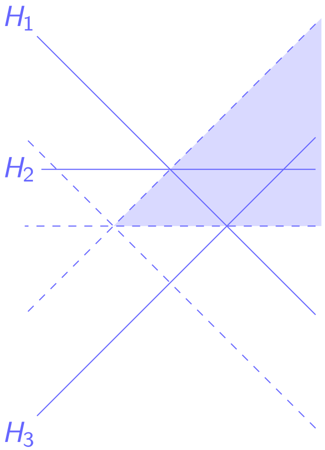




The Poincaré polynomial of this cone is $\text{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

Example Cont'd



On the previous slide, we saw that $\mathrm{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

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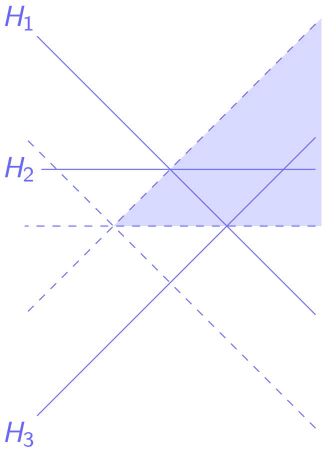

On the previous slide, we saw that $\mathrm{Poin}(\mathcal{C}, t) = 1 + 3t + t^2$.

The coefficients are the $n = 3$ **Narayana numbers**

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

which refine the Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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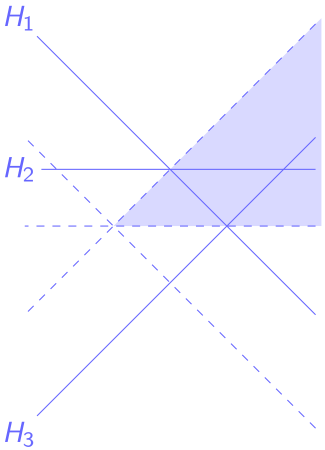

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This isn't a coincidence!

Shi Arrangements

Shi Arrangements

based on joint work with **Christian Stump**

For the experts: I'm going to tell you a "Type A" story, but this will hold for any irreducible crystallographic root system.

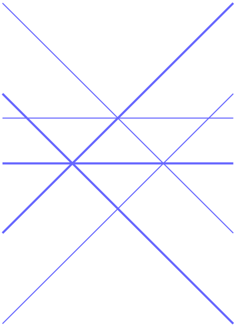
What is the Shi arrangement?

The (Type A) **Shi arrangement** has hyperplanes

$$H_{i,j,k} = \{x \in \mathbb{R}^n \mid x_i - x_j = k\}$$

for $i < j \in [n] := \{1, 2, \dots, n\}$ and $k = 0, 1$.

Below is an affine slice of the Shi arrangement for $n = 3$.



Weyl Cones

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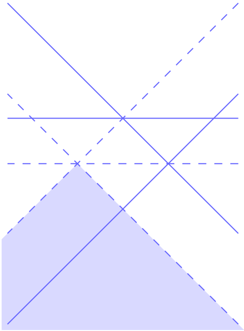
Below is an affine slice of the Shi arrangement for $n = 3$.

A **Weyl cone** of the Shi arrangement, is a cone which arises as a region of the **reflection subarrangement**

$$\{H_{i,j,0} \mid i < j \in [n]\}.$$

One Weyl cone is shaded on the right.



A Correspondence Between Cones and Permutations

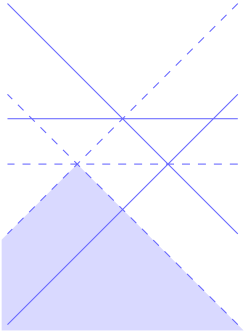
On the previous slide, saw that a Weyl cone of the Shi arrangement is a cone which arises as a region of the reflection subarrangement whose hyperplanes are

$$H_{i,k,0} = \{x \in \mathbb{R}^n \mid x_i - x_j = 0\}.$$

The regions $\sigma\mathcal{C}$ of this arrangement are in bijection with elements of the symmetric group $\sigma \in \mathfrak{S}_n$ via

$$x \in \sigma\mathcal{C} \iff x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n}$$

The $\sigma = 231$ cone is shaded on the left.



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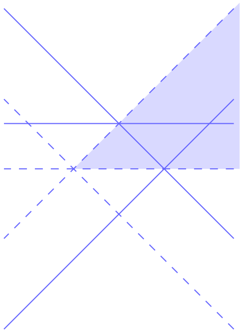
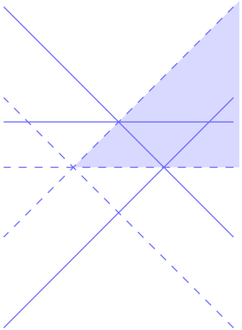
The $\sigma = 123$ cone is shaded on the left.



Catalan Numbers and Region Counts

Theorem (Athanasiadis)

When $\sigma = 12 \cdots n$ is the identity element of $\mathfrak{S}_n$, the number of regions in $\sigma\mathcal{C}$ is the nth Catalan number C_n.



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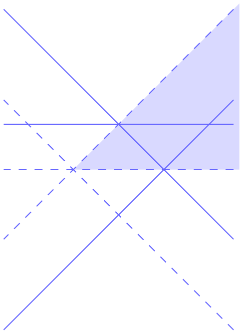

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Theorem ((DB)S, 22+)

When $\sigma = 12\cdots n$ is the identity element of $\mathfrak{S}_n$, the Whitney numbers of $\sigma\mathcal{C}$ are the Narayana numbers $N(n,k)$.

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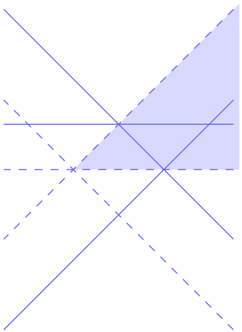

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This is a special case of one of our results, which describes all intersection posets of all Weyl cones. Unfortunately it requires more Coxeter theory than we have time for today.

A Note on the Proof

- The proof of our theorem uses “elemetentary” techniques and proves something stronger.
- The general theorem describes the poset of intersections via the antichains of a certain poset E_σ, which specializes to the *root poset* when σ is the identity.
- The numerical statement can be proved using commutative algebra... and that's what I want to tell you about for the remainder of this talk!

Two Rings

Warning: This section uses several terms that I haven't defined. Some useful references:

- Section 1 of "Gröbner Bases and Convex Polytopes" by Sturmfels
- Chapter 2 of "Ideals, Varieties, and Algorithms" by Cox, Little, O'Shea

The (very minor) extension to polynomial rings over $\mathbb{Z}$ is given in arXiv:2104.02740.

A Ring from Order Ideals

Let P be a poset and $J(P)$ its collection of order ideals ($=$ down-sets).

Definition

The **free poset ring** of P is the set of maps $f : J(P) \to \mathbb{Z}$ with pointwise addition and multiplication.

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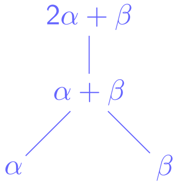

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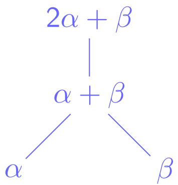

This poset has 6 antichains $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha, \beta\}$, $\{\alpha + \beta\}$, and $\{2\alpha + \beta\}$. An element of the free poset ring is an assignment of an integer weight to each of these order ideals.

A Ring from Order Ideals

Let $FP(P)$ be the free poset ring of P.

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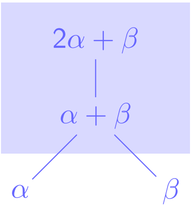

If I contains an element in the shaded region, then $x_{\alpha+\beta}(I) = 1$.

Theorem

This ring is generated by Heaviside functions, i.e.

$$x_i(I) = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{else} \end{cases}$$

for $i \in P$ and $I \in J(P)$.

A Ring from Order Ideals

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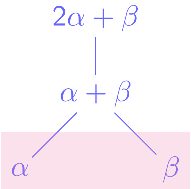

If I is contained in the shaded region, then $1 - x_{\alpha+\beta}(I) = 1$. The empty ideal, for example, is contained in the shaded region.

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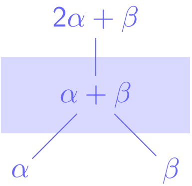

If I is generated by elements in the shaded region, then

$$x_{\alpha+\beta}(1-x_{2\alpha+\beta})(I)=1$$

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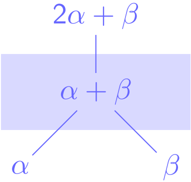

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$$x_i(I)=\begin{cases}1 & \text{if } i\in I\\ 0 & \text{else}\end{cases}$$

for $i \in P$ and $I \in J(P)$. In particular, the following map is surjective

$$\mathbb{Z}[e_i \mid i\in P]\to FP(P)$$
$$e_i\mapsto x_i$$

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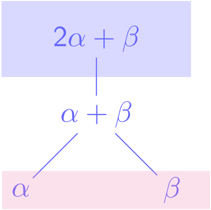
Let $FP(P)$ be the free poset ring of P.



Its not hard to see why $(1-x_{\alpha+\beta})x_{2\alpha+\beta}=0$

Theorem

This ring is generated by Heaviside functions, i.e.

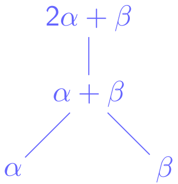
$$x_i(I)=\begin{cases}1 & \text{if } i\in I\\ 0 & \text{else}\end{cases}$$

for $i\in P$ and $I\in J(P)$. In particular, the following map is surjective

$$\mathbb{Z}[e_i \mid i\in P]\to FP(P)$$
$$e_i\mapsto x_i$$

and its kernel contains $\mathcal{G}=\{e_j(1-e_i)\mid i\leq_P j\}$.

A Ring from Order Ideals



This poset has 6 antichains $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\alpha,\beta\}$, $\{\alpha+\beta\}$, and $\{2\alpha+\beta\}$.

The associated graded has Hilbert series

$$1+4t+t^2.$$

Theorem

For any degree monomial order, the free poset ring $FP(P)$ and its associated graded (w.r.t. the degree filtration) have presentations

$$FP(P) \cong \mathbb{Z}[e_i \mid i \in P]/(\mathcal{G})$$
$$\mathfrak{gr}FP(P) \cong \mathbb{Z}[e_i \mid i \in P]/(in_{\deg}\mathcal{G})$$

and moreover

$$\mathrm{Hilb}(\mathfrak{gr}FP(P);t) = \sum_{\substack{A\subseteq P\\ \text{antichain}}} t^{\#A}.$$

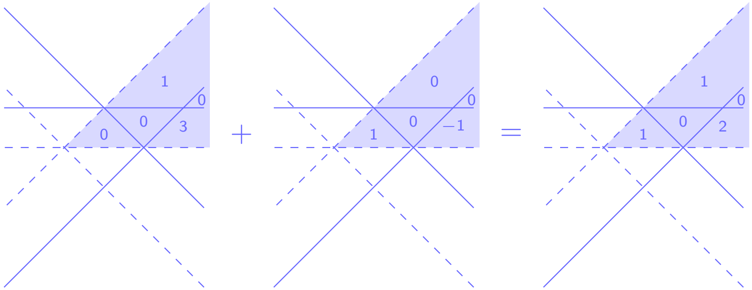
A version of this theorem was proved by Chapoton.

A Ring from Regions

Let $\mathcal{A}$ be an arrangement and $\mathcal{C}$ a cone with regions $\mathcal{R}(\mathcal{C})$.

Definition

The Varchenko-Gel'fand ring of $\mathcal{C}$ is the set of maps $f : \mathcal{R}(\mathcal{C}) \to \mathbb{Z}$ with pointwise addition and multiplication.



A Ring from Regions

Let $\mathcal{A}$ be an arrangement and $\mathcal{C}$ a cone with regions $\mathcal{R}(\mathcal{C})$.

Theorem (DB, 21)

One can explicitly describe a collection of polynomials $\mathcal{G} \subseteq \mathbb{Z}[e_H \mid H \in \mathcal{A}]$ such that for any "compatible" monomial order

$$VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(\mathcal{G})$$
$$\mathfrak{gr}VG(\mathcal{C}) \cong \mathbb{Z}[e_H \mid H \in \mathcal{A}]/(in_{\deg}\mathcal{G})$$

where $\mathfrak{gr}VG(\mathcal{C})$ is the associated graded of $VG(\mathcal{C})$ with respect to a certain filtration. In particular, the Hilbert series is

$$\text{Hilb}(\mathfrak{gr}VG(\mathcal{C}); t) = \text{Poin}(\mathcal{C}, t).$$

The $\mathcal{C} = V$ case was first proved by Varchenko and Gel'fand.

The Two Rings Together

Let $Shi(n)$ be the nth Shi arrangement,

$\sigma \in \mathfrak{S}_n$,

σC a Weyl cone, and

E_σ the poset we introduced (but didn't define) earlier.

Upshot

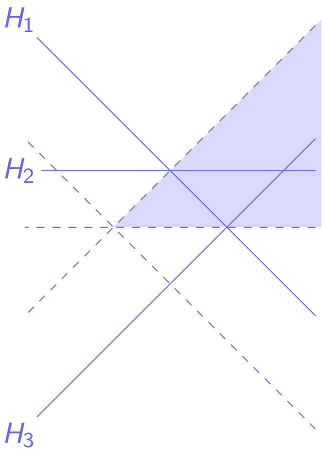
Combining the previous statements gives $\mathbb{Z}$-algebra isomorphisms

$$VG(\sigma C) \cong FP(E_\sigma)$$
$$\mathfrak{gr} VG(\sigma C) \cong \mathfrak{gr} FP(E_\sigma),$$

and in particular

$$\mathrm{Poin}(\sigma C, t) = \mathrm{Hilb}(\mathfrak{gr} VG(\sigma C); t) = \sum_{\substack{\text{anitchains} \\ A \subseteq E_\sigma}} t^{\#A}.$$

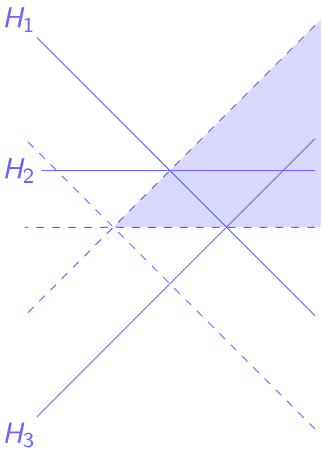
Back to Narayna Numbers



On the previous slide, we saw that

$$\mathrm{Poin}(\sigma C, t) = \sum_{\substack{\text{anitchains} \\ A \subseteq E_\sigma}} t^{\# A}.$$

Back to Narayna Numbers



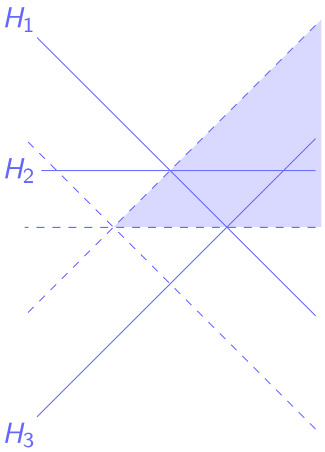
On the previous slide, we saw that

$$\mathrm{Poin}(\sigma C, t) = \sum_{\substack{\text{anitchains} \\ A \subseteq E_\sigma}} t^{\#A}.$$

When $\sigma = 12\cdots n$, the poset E_σ is the root poset Φ^+. One definition of the Narayana numbers is

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \# \left\{ \begin{matrix} \text{antichains of } \Phi^+ \\ \text{of cardinality } k \end{matrix} \right\}$$

Back to Narayna Numbers



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which refine the Catalan numbers

$$C_n = \# \left\{ \text{antichains of } \Phi^+ \right\}.$$

A Question for the Audience

Have you seen the free poset ring before?

Thank you for your attention!

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