Cones of Hyperplane Arrangements and the Varchenko-Gel'fand Ring

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Hyperplane Arrangements & Oriented Matroids

Linear Systems in $\mathbb{R}^d$

In linear algebra courses, we teach our students how to solve linear systems like this one

$$\begin{aligned} x - y &= 0 \\ y - z &= 0 \\ x - z &= 0. \end{aligned}$$

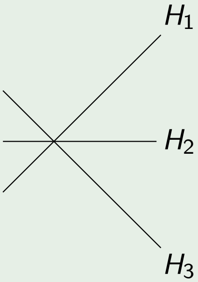
That is, we ask them to find all points $(x, y, z) \in \mathbb{R}^3$ which are simultaneously contained in these three planes.

Today: We'll study collections of (hyper)planes in $\mathbb{R}^d$.

Arrangements of Hyperplanes

- A *hyperplane* is a ~~affine~~ linear subspace of codimension 1.
- A distinct collection of finitely-many hyperplanes is an *arrangement*.
- There are many ways to study arrangements discretely. Today we'll focus on
  - regions (= open, connected components of the complement), and
  - intersections (= nonempty intersections of some of the hyperplanes).

Example



This arrangement has 6 regions and the set of intersections is

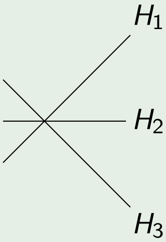
$$\mathbb{R}^2, \; H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$

A Partially-Ordered Set

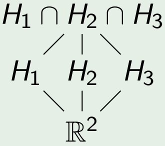
Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ can be *partially-ordered* by reverse inclusion.
- On the right, we show an example of an arrangement $\mathcal{A}$, together with the *Hasse diagram* of its *poset* of intersections $\mathcal{L}(\mathcal{A})$.

Example



The poset of intersections is

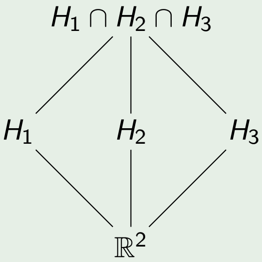


Möbius function

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- Every poset comes equipped with a *Möbius function* $\mu(Y, X)$ for $Y <_P X$.
- Here, we will be interested in the Möbius function of *lower intervals* $[\mathbb{R}^d, X]$ for $X \in \mathcal{L}(\mathcal{A})$.
- We'll define $\mu(\mathbb{R}^d, X)$ by example on the right side.

Example



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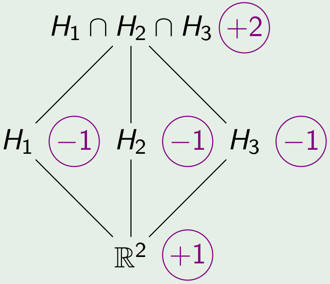
Example

For each X, we give the value of $\mu(\mathbb{R}^d, X)$ beside X.



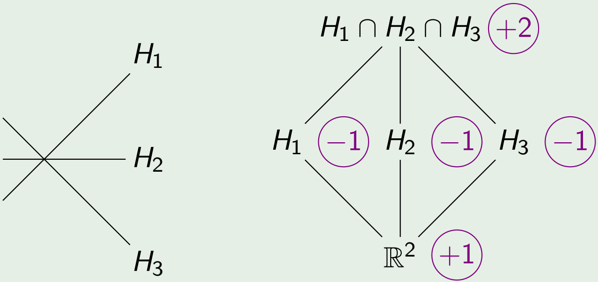
Zaslavsky's Theorem

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$.

Theorem (Zaslavsky)

$$\#\mathcal{R}(\mathcal{A}) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)|$$

Example



Zaslavsky's theorem: $1 + 3(1) + 2 = 6$.

Example: Braid Arrangement

Let A_{d-1} be the arrangement with the $\binom{d}{2}$ hyperplanes

$$H_{ij} := \{\mathbf{x} \in \mathbb{R}^d \mid x_i - x_j = 0\} \qquad \text{for } i < j \in [d] := \{1, \ldots, d\}.$$

- The regions of the Braid Arrangement A_{d-1} are in bijection with the permutations $\mathfrak{S}_d$ of d objects.
- The poset of (nonempty) intersections is isomorphic (as a poset) to the poset of (set) partitions Π_d of $[d]$ ordered by refinement.

Zaslavsky's theorem says

$$\#\mathfrak{S}_d = \sum_{\pi \in \Pi_d} \prod_{B \in \pi} (\#B - 1)!$$

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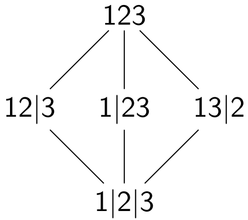

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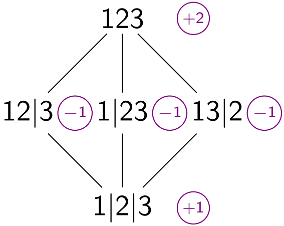


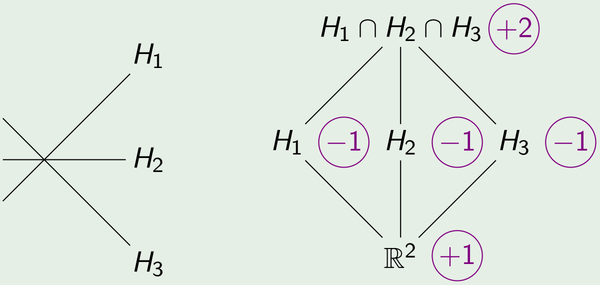
Zaslavsky's theorem says that $1 + 3(1) + 2 = 6 = 3!$.

The Poincaré Polynomial

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$ and intersections $\mathcal{L}(\mathcal{A})$. Define the *Poincaré polynomial* of $\mathcal{A}$ by

$$\operatorname{Poin}(\mathcal{A}, t) = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V, X)| t^{d-\dim(X)}.$$

Example



The Poincaré polynomial of this arrangement is $\operatorname{Poin}(\mathcal{A}, t) = 1 + 3t + 2t^2$.

Example: Braid Arrangement

Let A_{d-1} be the arrangement with the $\binom{d}{2}$ hyperplanes

$$H_{ij} := \{\mathbf{x} \in \mathbb{R}^d \mid x_i - x_j = 0\} \qquad \text{for } i < j \in [d] := \{1, \dots, d\}.$$

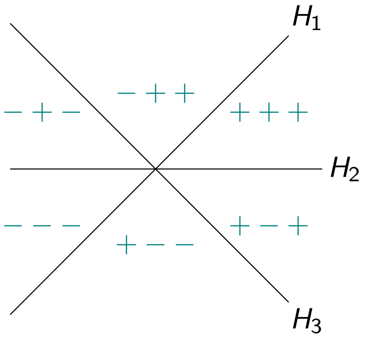
The Poincaré polynomial of A_{n-1} is

$$\operatorname{Poin}(A_{n-1}, t) = \sum_{k \geq 1} \#\{\sigma \in \mathfrak{S}_d \mid \sigma \text{ has } k \text{ cycles}\}\ t^k.$$

Oriented Matroids

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$.

- On the right, we show an example of an arrangement $\mathcal{A}$, together with a labelling of the regions by *signed sets*.
- Not shown: this labelling works for the lower-dimensional faces as well!
- These signed sets satisfy the *(co)vector axioms* and thus define an oriented matroid.



I won't formally state the (co)vector axioms in this talk, but we can go over them at the end if there is interest!

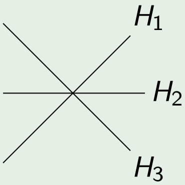
(Unoriented) Matroids

Chris and Eleanore recently discussed (unoriented) matroids.

- Every oriented matroid has an underlying unoriented matroid. Instead of looking at signed sets $X = (X^+, X^-)$, look at $\underline{X} = X^+ \cup X^-$. [1]
- There is also a notion of oriented matroid duality. In our setting, the dual of the "facial oriented matroid" encodes (signed) linear dependencies among the normal vectors to the hyperplanes.

Example

Normal vectors: $v_1 = (1, -1), v_2 = (0, 1)$, and $v_3 = (1, 1)$.



- The linear dependence $v_1 + 2v_2 - v_3 = (0, 0)$ corresponds to the signed set $+ + -$
- The linear dependence $-v_1 - 2v_2 + v_3 = (0, 0)$ corresponds to the signed set $- - +$

[1] The opposite is not true, see Bland-Las Vergnas "Orientability of Matroids."

Useful Facts About Oriented Matroids

Consider an oriented matroid on the ground set $[n]$.

- The signed sets associated to minimal dependencies among the normal vectors are called the *signed circuits*.
- If an oriented matroid has signed circuits $\mathbf{C}$, then the unsigned circuits of the underlying unoriented matroid are $\underline{\mathbf{C}} := \{\underline{C} \mid C \in \mathbf{C}\}$ We can *break* an unsigned circuit by removing the smallest entry of $\underline{C}$.
- Any subset $N \subseteq [n]$ NOT containing a broken circuit is a *no broken circuit set* of the oriented matroid.

Example

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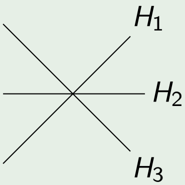


- Signed circuits: $++-$, $--+$
- Unsigned circuit: $\{1, 2, 3\}$
- No broken circuit sets: $\emptyset$, 1, 2, 3, 12, 13

The Varchenko-Gel'fand Ring

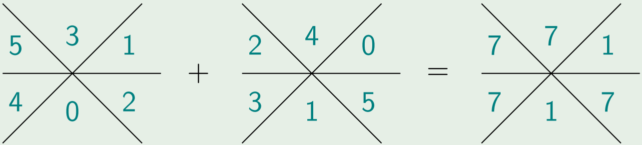
A Ring from Regions

Let $\mathcal{A}$ be an arrangememnt of hyperplanes in $\mathbb{R}^d$ with regions $\mathcal{R}(\mathcal{A})$.

Definition

The Varchenko-Gel'fand ring of $\mathcal{A}$ is the set of maps $f : \mathcal{R}(\mathcal{A}) \to \mathbb{Z}$ with pointwise addition and multiplication.

Example



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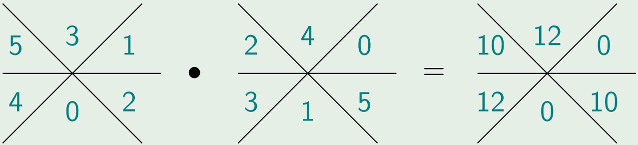


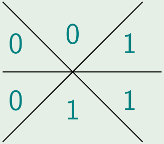
Generators for the Varchenko-Gel'fand ring

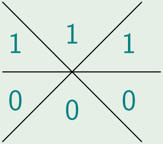
Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$.
Choose a set of normal vectors such that n_H is the normal vector to $H \in \mathcal{A}$. Define a *Heaviside function*

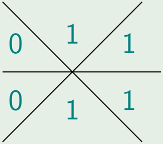
$$x_H(v) = \begin{cases} 1 & \text{if } \langle v, n_H \rangle > 0 \\ 0 & \text{else.} \end{cases}$$

We can define this instead on regions, by choosing a representative point $v \in R$ for each region and defining $x_H(R) = x_H(v)$.

Example

$x_1 =$ 

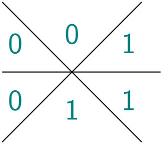
$x_2 =$ 

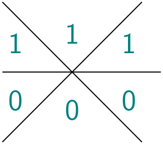
$x_3 =$ 

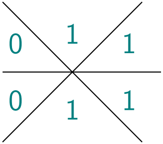
Generators for the Varchenko-Gel'fand ring

Lemma

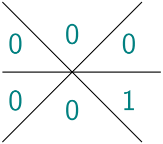
Together with 1*, these Heaviside functions generate the Varchenko-Gel'fand ring as a* $\mathbb{Z}$*-algebra.*

$x_1 =$ 

$x_2 =$ 

$x_3 =$ 

Let's write out the following element as a polynomial in these Heaviside functions.



Generators for the Varchenko-Gel'fand ring

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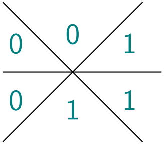
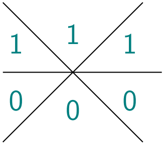
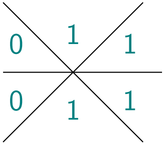
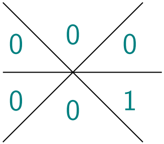
$x_1 =$ 

$x_2 =$ 

$x_3 =$ 

Let's write out the following element as a polynomial in these Heaviside functions.

$x_1 x_3 (1 - x_2) =$ 

Example: Braid Arrangement

Definition

The Varchenko-Gel'fand ring of $\mathcal{A}$ is the set of maps $f : \mathcal{R}(\mathcal{A}) \to \mathbb{Z}$ with pointwise addition and multiplication.

Lemma

These Heaviside functions generate the Varchenko-Gel'fand ring as a $\mathbb{Z}$-algebra.

Let A_{d-1} be the Braid Arrangement. Regions are in bijection with permutations of d objects, and moreover

$$x_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j) \\ 0 & \text{else} \end{cases}$$

for $i < j$ in $[n]$. Our lemma says that we can describe $\sigma \in \mathfrak{S}_d$ by its inversion set.

A Filtration by Degree

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$.

- We just saw that the Varchenko-Gel'fand ring is generated by Heaviside functions defined by the hyperplanes of $\mathcal{A}$.
- It also has a filtration $\mathcal{F} : F_0 \subseteq F_1 \subseteq \cdots$ by degree, i.e., the collection of additive groups

$$\begin{aligned} F_0 &= \mathbb{Z} - \text{span}\{1\} \\ F_1 &= \mathbb{Z} - \text{span}\{1\} \cup \{x_H \mid H \in \mathcal{A}\} \\ &\vdots \\ F_i &= \mathbb{Z} - \text{span}\{\text{monomials of degree} \leq i\}. \end{aligned}$$

- The *associated graded ring* is $\mathcal{VG}(\mathcal{A}) = \bigoplus_{i \geq 0} F_i / F_{i-1}$ and its *Hilbert series* is

$$\text{Hilb}(\mathcal{VG}(\mathcal{A}), t) = \sum_{i \geq 0} \text{rk}_{\mathbb{Z}}(F_i / F_{i-1})\ t^i.$$

Some Results

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$ with intersection poset $\mathcal{L}(\mathcal{A})$.

Theorem (Varchenko-Gel'fand)

Each graded component F_i/F_{i-1} of $\mathcal{VG}(\mathcal{A})$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis indexed by the no broken circuit sets of the arrangement.

Theorem (Rota)

For $X \in \mathcal{L}(\mathcal{A})$, we have

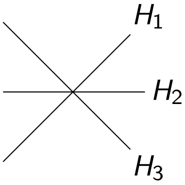
$$|\mu(\mathbb{R}^d, X)| = \#\{\textit{no broken circuit sets whose join is } X\}.$$

Combining these theorems gives

$$\mathrm{Hilb}(\mathcal{VG}(\mathcal{A}), t) = \mathrm{Poin}(\mathcal{A}, t).$$

Example

Consider the arrangement in $\mathbb{R}^2$ with normal vectors $v_1 = (1, -1)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$ (drawn below, left).



- Signed circuits: $++-$, $--+$
- Unsigned circuit: $\{1, 2, 3\}$
- No broken circuit sets: $\emptyset$, 1, 2, 3, 12, 13

Varchenko-Gel'fand showed that

$$\mathcal{VG}(\mathcal{A}) \cong \mathbb{Z} \cdot \{1\} \oplus \mathbb{Z} \cdot \{x_1,\ x_2,\ x_3\} \oplus \mathbb{Z} \cdot \{x_1x_2,\ x_1x_3\}$$

where $\mathbb{Z} \cdot \{-\}$ denotes the $\mathbb{Z}$-span of $-$. Then the Hilbert series is

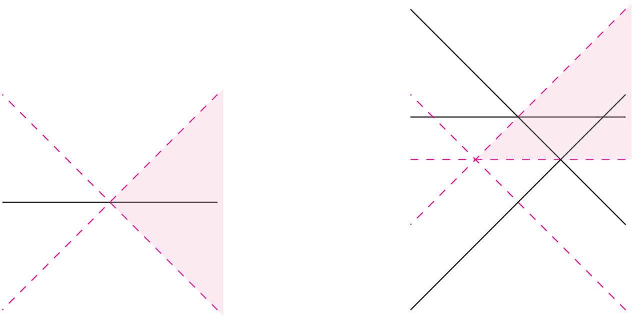
$$\mathrm{Hilb}(\mathcal{VG}(\mathcal{A}), t) = 1 + 3t + 2t^2$$

which matches the Poincaré polynomial we computed earlier.

Instead of looking at arrangements $\mathcal{A}$, look at *cones*.

- A cone $\mathcal{K}$ of an arrangement $\mathcal{A}$ is an intersection of (open) halfspaces defined by some of the hyperplanes of $\mathcal{A}$.
- Cones are interesting in the theory of arrangements, as they unify the theory of *central* and *affine* arrangements while generalizing both.

Below are two examples of a cones in arrangements.



Cones of Arrangements

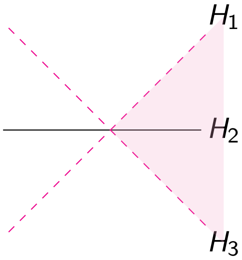
Let $\mathcal{K}$ be a cone of an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$.

- The regions of a cone $\mathcal{K}$ are the regions of the arrangement which lie inside the open cone.
- The intersections of such a cone $\mathcal{L}(\mathcal{K})$ are the intersections X of the arrangement such that $X \cap \mathcal{K} \neq \emptyset$.
- It turns out that $\mathcal{L}(\mathcal{K})$ is an *order ideal* of $\mathcal{L}(\mathcal{A})$, so that it makes sense to define

$$\mathrm{Poin}(\mathcal{K}, t) = \sum_{X \in \mathcal{L}(\mathcal{K})} |\mu(\mathbb{R}^d, X)| t^{d - \dim(X)}.$$

Example: Cones of Arrangements

Consider the shaded cone $\mathcal{K}$ in the arrangement in $\mathbb{R}^2$ (drawn below).



- This cone has two chambers.
- It has intersections $\mathbb{R}^2$, H_2.
- Its Poincaré polynomial is $\mathrm{Poin}(\mathcal{K}, t) = 1 + t$.

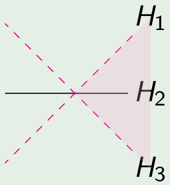
Cones of Arrangements

Let $\mathcal{K}$ be a cone of an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ and let $NBC(\mathcal{A})$ denote the set of no broken circuit monomials of the (unoriented) matroid of $\mathcal{A}$.

- Say that $N \in NBC(\mathcal{A})$ is a $\mathcal{K}$-no-broken-circuit set if the intersection $\bigcap_{i \in N} H_i$ has a nonempty intersection with $\mathcal{K}$.

Example

Consider the arrangement in $\mathbb{R}^2$ with normal vectors $v_1 = (1, -1)$, $v_2 = (0, 1)$, and $v_3 = (1, 1)$ (drawn below, left).



- Signed circuits: $+ + -$, $- - +$
- Unsigned circuit: $\{1, 2, 3\}$
- No broken circuit sets: $\emptyset$, 1, 2, 3, 12, 13
- $\mathcal{K}$-no-broken-circuits: $\emptyset$, 2

Generalized Statement of Results

Let $\mathcal{K}$ be a cone of an arrangement $\mathcal{A}$ in $\mathbb{R}^d$.

Theorem (DB)

Each graded component F_i/F_{i-1} of $\mathcal{VG}(\mathcal{K})$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis indexed by $\mathcal{K}$-no broken circuit sets of the arrangement. Moreover

$$\mathrm{Hilb}(\mathcal{VG}(\mathcal{K}), t) = \mathrm{Poin}(\mathcal{K}, t).$$

This theorem is a corollary to the main theorem of the paper. The main theorem defines a certain collection of polynomials $\mathcal{G}$ and uses them to give presentations for the Varchenko-Gel'fand ring and its associated graded:

$$VG(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots, e_n]/(\mathcal{G})$$
$$\mathcal{VG}(\mathcal{K}) \cong \mathbb{Z}[e_1, \ldots, e_n]/(in_{deg}\mathcal{G})$$

where $in_{deg}\mathcal{G}$ denotes the top-degree form of $g \in \mathcal{G}$.

I won't state these precisely in this talk, but we can discuss them at the end, if there is interest!

A Consequence of the Approach

Warning: This section uses several terms that I haven't defined. Some useful references:

- Section 1 of "Gröbner Bases and Convex Polytopes" by Sturmfels
- Chapter 2 of "Ideals, Varieties, and Algorithms" by Cox, Little, O'Shea

Working over a Field

Let $\mathcal{K}$ be a cone of an arrangement $\mathcal{A}$ in $\mathbb{R}^d$. The same arguments work over a field $\mathbb{F}$ (and in fact they're even easier!)

Corollary (DB)

Using the same collection of polynomials $\mathcal{G}$ as in the integer case, we have

$$VG(\mathcal{K}) \cong \mathbb{F}[e_1, \dots, e_n]/(\mathcal{G})$$
$$\mathcal{VG}(\mathcal{K}) \cong \mathbb{F}[e_1, \dots, e_n]/(in_{deg}\mathcal{G})$$

where $\mathcal{G}$ is a Gröbner basis for $(\mathcal{G})$ (we assume that the given monomial order $\prec$ satisfies $e_1 \prec \cdots \prec e_n$).

In the next few slides, we'll look at one consequence of this extension.

Quadratic Gröbner bases

Let R be a *commutative standard graded* $\mathbb{F}$*-algebra*, i.e. $R \cong \mathbb{F}[e_1, \ldots, e_n]/I$ where I is a homogeneous ideal and each e_i has degree exactly 1. Suppose

$$F_\bullet : \cdots \xrightarrow{\varphi_3} R^{\beta_2} \xrightarrow{\varphi_2} R^{\beta_1} \xrightarrow{\varphi_1} R \longrightarrow \mathbb{F}$$

is a *minimal free resolution* of the R-module $\mathbb{F} = R/R_+$ where R_+ is the maximal homogeneous ideal, consisting of all elements of positive degree.

Definition

R is *Koszul* if the nonzero entries of each φ_i matrix are homogeneous of degree 1.

Theorem

Let I be a homogeneous ideal in $\mathbb{F}[e_1, \ldots, e_n]$. Suppose there exists a monomial order $\prec$ and quadratic Gröbner basis $\mathcal{G}$ generating I. Then $R = \mathbb{F}[e_1, \ldots, e_n]/I$ is Koszul.

Quadratic Gröbner bases

Theorem

Let I be a homogeneous ideal in $\mathbb{F}[e_1, \ldots, e_n]$. Suppose there exists a monomial order $\prec$ and quadratic Gröbner basis $\mathcal{G}$ generating I. Then $R = \mathbb{F}[e_1, \ldots, e_n]/I$ is Koszul.

Takeaway: If there is a family of arrangements (or cones) for which $in_{deg}(\mathcal{G})$ is quadratic, then this theorem tells us that $\mathcal{VG}(\mathcal{A})$ will be Koszul.

Quadratic Gröbner bases

One such family are *supersolvable arrangements*, of which the Braid Arrangement A_{d-1} is an example.

Theorem (Björner-Ziegler)

Let $\mathcal{A}$ be a supersolvable arrangement and M its underlying (unoriented) matroid. Then the minimal broken circuits of M (under inclusion) have cardinality 2.

Its not immediate, but one can use this Björner-Ziegler result in conjunction with the quadtratic Gröbner basis result to show:

Theorem (DB)

If $\mathcal{A}$ is a supersolvable arrangement, then $\mathcal{VG}(\mathcal{A})$ is Koszul.

Questions for the Audience

- (for early graduate students) Are there places in your research where a proof had implications not directly implied by the theorem you wanted to prove?
- (for everyone) The final theorem in the talk was a commutative version of a theorem of Irena Peeva (she proved the equivalent statement for the Orlik-Solomon algebra). Are there other statements from the Orlik-Solomon algebra, which have analogues for the Varchenko-Gel'fand ring?
- (for everyone) In the Braid Arrangement, cones correspond to posets on d. There are "poset cones" $\mathcal{K}_P$ in the Braid Arrangement for which $\mathcal{VG}(\mathcal{K}_P)$ is not Koszul. Is there some interesting family of posets for which $\mathcal{VG}(\mathcal{K}_P)$ is Koszul?[2]

[2] If you find one, I have a follow-up question! (But its too technical for today!)

Thank you for listening!

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