The Poincaré-extended **ab**-index

Galen Dorpalen-Barry
joint with Joshua Maglione and Christian Stump

arXiv:2301.05904 + JLMS (2025) + FPSAC extended abstract (2024)

ESI Workshop: Recent Perspectives on Non-crossing Partitions
February 16, 2025

TEXAS A&M UNIVERSITY

Outline

1. Motivation
2. Main Objects
3. Results

Motivation

Igusa Zeta Functions & their Poles

- Grunewald, Sega, and Smith (1988) - define “subgroup zeta function” of a finitely-generated group
- Du Sautoy and Grunewald (2000) - general method to compute zeta functions
- Maglione and Voll (2023) - in the case where the input polynomial is a product of linear things:
  - Calculation only depends on the **intersection poset** of the corresponding **hyperplane arrangement**
  - Result has only one pole: $t = 1$
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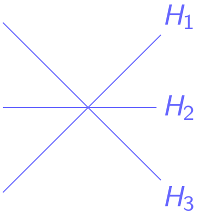
A Key Tool: a combinatorially-defined polynomial that says something mysterious about noncrossing partitions.

Main Objects

Arrangements of Hyperplanes in $\mathbb{R}^d$

- A **hyperplane** is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.



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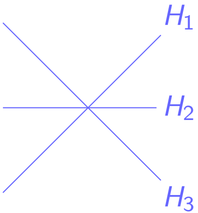

Today we'll focus on **intersections** (= nonempty intersections of some of the hyperplanes).

Arrangements of Hyperplanes in $\mathbb{R}^d$

The set of intersections of this arrangement is

$$\mathbb{R}^2,\ H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$

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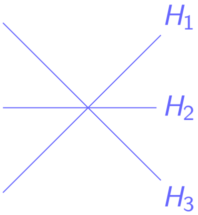

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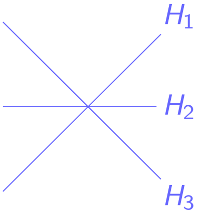
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of **regions** of the arrangement.



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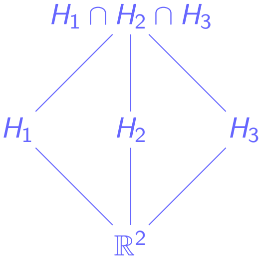

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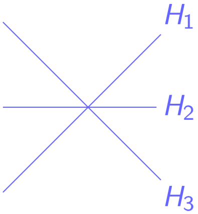

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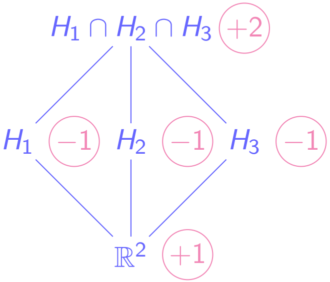

The Poincaré Polynomial of a Poset

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Definition

The **Poincaré polynomial** of $\mathcal{L}$ is

$$\mathrm{Poin}(\mathcal{L}; y) = \sum_{x \in \mathcal{L}} |\mu(\hat{0}, x)| \; y^{\mathrm{codim}(x)},$$

where $\mathrm{codim}(x)$ denotes the codimension of x.

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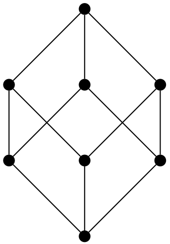
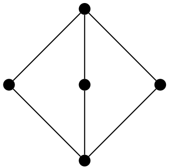
Tells us the Hilbert series of the Orlik-Solomon Algebra and Varchenko-Gelfand ring.



$1 + 3y + 3y^2 + y^3$



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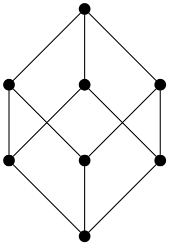
Note. We can define the Poincaré polynomial for any *graded poset*.



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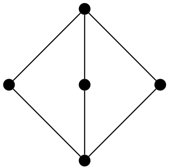

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The Poincaré-extended **ab**-index

Definition

The **(Poincaré-)extended ab-index** of $\mathcal{L}$ is

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } \mathcal{L}\setminus\{\hat{1}\}} \mathrm{Poin}(\mathcal{L}, \mathcal{C}, y)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

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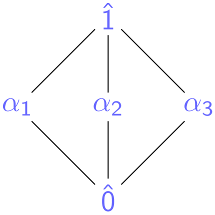

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0}<\alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2\end{aligned}$$

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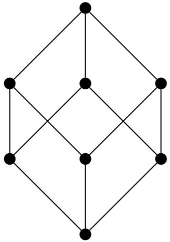

For the poset on the left:

$$\begin{aligned}\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) &= \mathbf{a}^3 + (3y+2)\mathbf{a}^2\mathbf{b} + (3y^2+6y+2)\mathbf{aba} \\ &+ (3y^2+3y+1)\mathbf{ab}^2 + (y^3+3y^2+3y)\mathbf{ba}^2 \\ &+ (2y^3+6y^2+3y)\mathbf{bab} + (2y^3+3y^2)\mathbf{b}^2\mathbf{a} \\ &+ y^3\mathbf{b}^3\,.\end{aligned}$$

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Let P be a graded poset.

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(\mathcal{L}; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! Before we get into the proof, let's look at where their conjecture comes from...

Motivation: Analytic Zeta Functions

Let $\mathcal{A}$ be a central hyperplane arrangement in a real vector space with intersection lattice $\mathcal{L}$.

Maglione–Voll prove that (after a change of variables) the **(coarse) analytic zeta function** of $\mathcal{A}$ is

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)\left(\frac{t}{1-t}\right)^{\#\mathcal{C}}.$$

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This is a bivariate version of the **analytic zeta function**.

A different bivariate specialization of their analytic zeta function recovers the celebrated **Motivic Zeta function** of a matroid given by Jensen–Kutler–Usatine.

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Putting all terms over the same denominator gives

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \frac{\mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\}, y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}}{(1-t)^{\mathrm{rank}(\mathcal{A})}}.$$

The numerator of this rational function is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\}, y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}.$$

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We can now state Maglione–Voll's conjecture more precisely:

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

Kühne–Maglione studied $Num_{\mathcal{A}}(1,t)$ as well, and conjectured that

$$\mathrm{Poin}(\mathcal{A},1)\cdot(1+t)^{\mathrm{rank}\mathcal{A}-1} \leq Num_{\mathcal{A}}(1,t).$$

We won't discuss it today, but our proof of the Maglione-Voll conjecture will give a proof of Kühne–Maglione's conjecture (almost) for free!

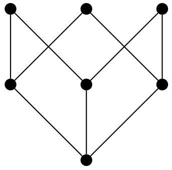
Results

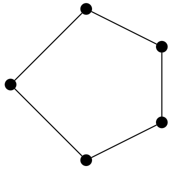
Graded Posets

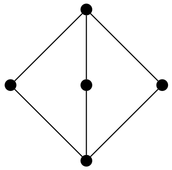
Let P be a poset with $\hat{0}$ and $\hat{1}$.

- A **chain** is a subset of the ground set which is totally ordered with respect to P.
- A chain $\mathcal{C} = C_1 < C_2 < \cdots C_n$ is **maximal** if C_i covers C_{i+1} for all $i = 1, \ldots, n-1$.
- P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.
- For $x, y \in P$, the **interval** between x and y is

$$[x, y] = \{z \mid x \leq z \leq y\}.$$





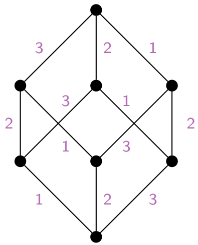


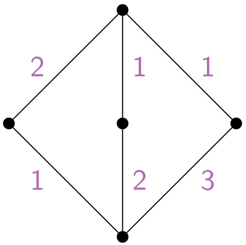
R-labelings

Let P be a graded poset, and let $\mathcal{E}(P) = \{(x, y) \mid x, y \in P,\ x \lessdot y\}$ denote the set of cover relations of P.

A labeling $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is an R-**labeling** if for every interval $[x, y]$, there is a unique maximal chain $\mathcal{M} = \{x = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = y\}$ such that the labels *weakly* increase, i.e.,

$$\lambda(C_{i-1}, C_i) \leq \lambda(C_i, C_{i+1}) \qquad \text{for } i = 2, \ldots k - 1.$$





R-labelings

Theorem (Björner, 1980)

Upper-semimodular, lower-semimodular, and supersolvable arrangements admit R-labelings.

Upshot: Geometric lattices always have R-labelings.

Surprise Bonus Upshot: The noncrossing partition lattices have R-labelings.

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2025)

The extended **ab***-index of* P *is*

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M}, E)} y^{\#E} \mathrm{mon}(\mathcal{M}, E)$$

where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chain and* E *is a subset of its edges.*

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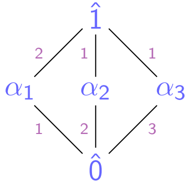
This immediately implies a Maglione–Voll's conjecture.

Surprise Bonus Upshot: This polynomial is nonnegative for noncrossing partition lattices.

Example

Computing $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ using the theorem instead of the definition.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

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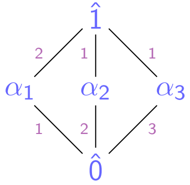

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Open Question: What do the coefficients of these polynomials tell us about noncrossing partitions?

Danke!

Selected References

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