The Poincaré-extended **ab**-index

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joint with Joshua Maglione and Christian Stump
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Outline

Setup + Motivation

Motivation 1: Counting Poles

- Zeta functions are used in group theory and can be used to capture discrete information about groups.

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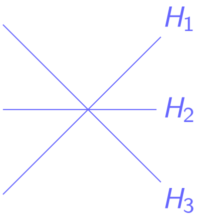
- Several people tried to prove their conjecture, and ended up making related conjectures.

  We proved the first conjecture and a related one.

Arrangements of Hyperplanes in $\mathbb{R}^d$

- A **hyperplane** is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.



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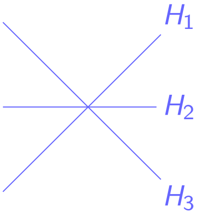

Today we'll focus on **intersections** (= nonempty intersections of some of the hyperplanes).

Arrangements of Hyperplanes in $\mathbb{R}^d$

The set of intersections of this arrangement is

$$\mathbb{R}^2,\ H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$

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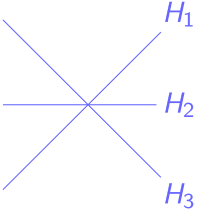

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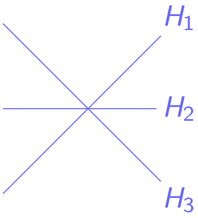
Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of **regions** of the arrangement.



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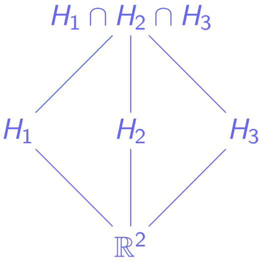

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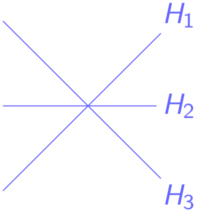

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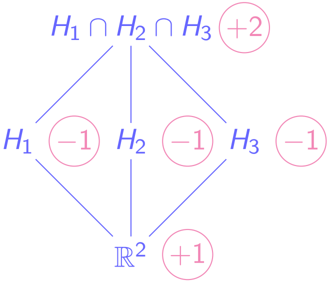

The Poincaré Polynomial of a Poset

Let $\mathcal{A}$ be a central, essential hyperplane arrangement and $\mathcal{L}$ its lattice of intersections.

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Definition

The **Poincaré polynomial** of $\mathcal{L}$ is

$$\mathrm{Poin}(\mathcal{L}; y) = \sum_{x \in \mathcal{L}} |\mu(\hat{0}, x)| \; y^{\mathrm{codim}(x)},$$

where $\mathrm{codim}(x)$ denotes the codimension of x.

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Similar to the characteristic polynomial $\chi(\mathcal{A}, t) = (-1)^{\mathrm{rank}(\mathcal{A})} T_{\mathcal{A}}(1-t, 0)$.

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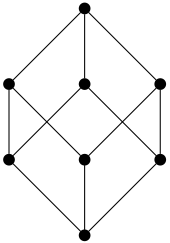
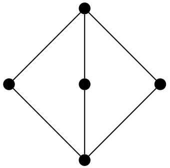
Tells us the Hilbert series' of the Orlik-Solomon and Cordovil algebras.



$1 + 3y + 3y^2 + y^3$



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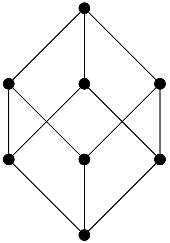
Note. We can define the Poincaré polynomial for any *graded poset*.



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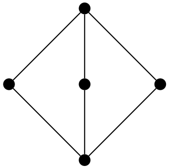

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Chain Poincaré Polynomials

$\mathcal{A}$: central, essential hyperplane arrangement
$\mathcal{L}$: lattice of intersections of $\mathcal{A}$
$\mathcal{C} = \{C_1 < \cdots < C_k\}$: chain of $\mathcal{L}$

The **chain Poincaré polynomial** of $\mathcal{C}$ is

$$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y) = \prod_{i=1}^{k} \mathrm{Poin}([C_i, C_{i+1}], y) \qquad \text{where } C_{k+1} = \hat{1}.$$

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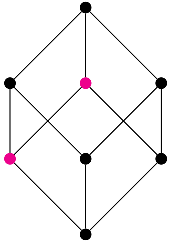

$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + y)^2$$

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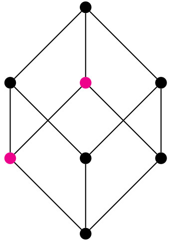

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$y = 1$ recovers the Bayer-Sturmfels relation: size of a fiber of a chain under the surjection $z : \Sigma^*(\mathcal{A}) \to \mathcal{L}$.

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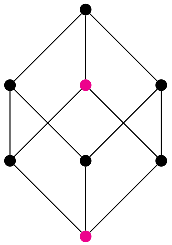

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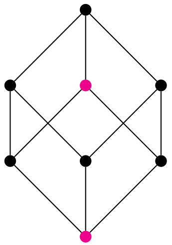

$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + 2y + y^2)(1 + y)$$

The Weight of a Chain

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If P is rank n (every maximal chain from $\hat{0}$ to $\hat{1}$ has length $n+1$) then the **weight** of a chain $\mathcal{C}$ is $\mathrm{wt}(\mathcal{C}) = w_1 \ldots w_n \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ where

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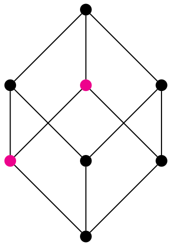

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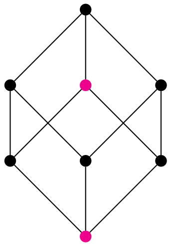

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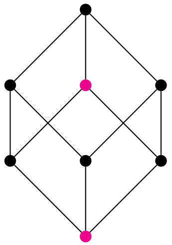

$$\mathrm{wt}(\mathcal{C}) = \mathbf{b}(\mathbf{a}-\mathbf{b})\mathbf{b}$$

The Poincaré-extended **ab**-index

Definition

The **(Poincaré-)extended ab-index** of $\mathcal{L}$ is

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } \mathcal{L}\setminus\{\hat{1}\}} \mathrm{Poin}(\mathcal{L}, \mathcal{C}, y)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

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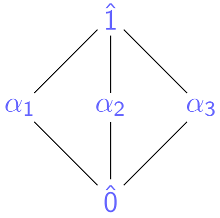

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0}<\alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2\end{aligned}$$

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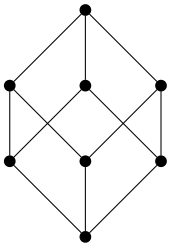

For the poset on the left:

$$\begin{aligned}\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \mathbf{a}^3 &+ (3y+2)\mathbf{a}^2\mathbf{b} + (3y^2+6y+2)\mathbf{aba} \\ &+ (3y^2+3y+1)\mathbf{ab}^2 + (y^3+3y^2+3y)\mathbf{ba}^2 \\ &+ (2y^3+6y^2+3y)\mathbf{bab} + (2y^3+3y^2)\mathbf{b}^2\mathbf{a} \\ &+ y^3\mathbf{b}^3\,.\end{aligned}$$

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then $\mathrm{ex}\Psi(\mathcal{L}; y, 1, t)$ has nonnegative coefficients.

This conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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This conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! Let's revisit where this conjecture comes from...

Motivation 2: Counting the Multiplicity of Poles

$\mathcal{A}$: central hyperplane arrangement in $V \cong \mathbb{R}^d$
$\mathcal{L}$: intersection lattice of $\mathcal{A}$

Maglione–Voll prove that the **(coarse) analytic zeta function** of $\mathcal{A}$ has the following combinatorial description:

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of} \mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C} \cup \{\hat{0}\}, y) \left(\frac{t}{1-t}\right)^{\#\mathcal{C}}.$$

This is a bivariate version of the **analytic zeta function**.

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A different bivariate specialization of their analytic zeta function recovers the celebrated **Motivic Zeta function** of a matroid given by Jensen–Kutler–Usatine.

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If we put everything over the same denominator, the numerator is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}.$$

Motivation 2: Counting the Multiplicity of Poles

The numerator of this rational function is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}\text{:chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \text{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\text{rank}(\mathcal{A})-\#\mathcal{C}}.$$

Conjecture (Maglione-Voll)

$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

- $\Rightarrow$ $t=1$ is not a root of $Num_{\mathcal{A}}(y,t)$
- $\Rightarrow$ Multiplicity of the pole $t=1$ in the rational expression is rank$(\mathcal{A})$

Motivation 2: Counting the Multiplicity of Poles

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Motivation 2: Counting the Multiplicity of Poles

The numerator of this rational function is

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Conjecture (Maglione-Voll)

$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

Kühne–Maglione studied $Num_{\mathcal{A}}(1,t)$ as well, and conjectured that

$$\mathrm{Poin}(\mathcal{A},1) \cdot (1+t)^{\mathrm{rank}\mathcal{A}-1} \leq Num_{\mathcal{A}}(1,t).$$

We won't discuss it today, but our proof of the Maglione-Voll conjecture will give a proof of Kühne–Maglione's conjecture (almost) for free!

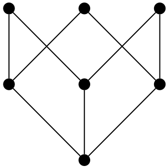
Combinatorial Machinery:
R-labeled Posets + Generalized Descent Sets

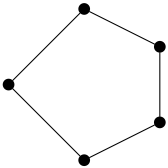
Graded Posets

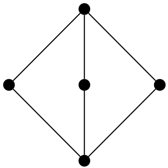
Let P be a poset with $\hat{0}$ and $\hat{1}$.

- A chain $\mathcal{C} = C_1 < C_2 < \cdots C_n$ is **maximal** if C_i covers C_{i+1} for all $i = 1, \ldots, n-1$.
- P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.
- For $x, y \in P$, the **interval** between x and y is

$$[x, y] = \{z \mid x \leq z \leq y\}.$$





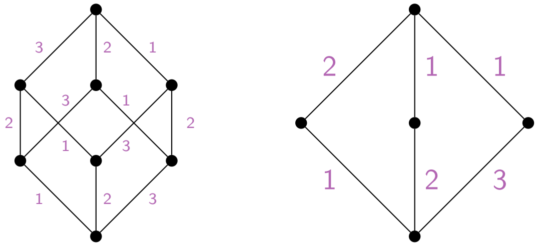


R-labelings

Let P be a graded poset, and let $\mathcal{E}(P) = \{(x, y) \mid x, y \in P,\ x \lessdot y\}$ denote the set of cover relations of P.

A labeling $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is an R-**labeling** if for every interval $[x, y]$, there is a unique maximal chain $\mathcal{M} = \{x = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = y\}$ such that the labels *weakly* increase, i.e.,

$$\lambda(C_{i-1}, C_i) \leq \lambda(C_i, C_{i+1}) \qquad \text{for } i = 2, \ldots k - 1.$$

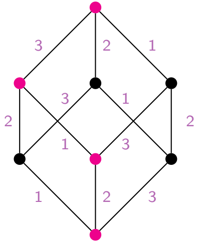


Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ.

Let $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ be a maximal chain of P. For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1}) .$$



This chain has a descent at position 1.

Descent Sets

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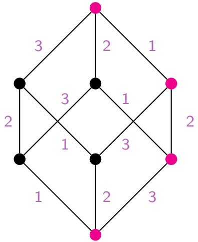


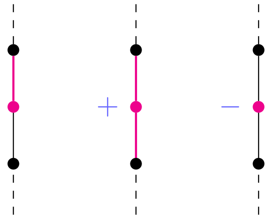
This chain has descents at positions 1 and 2.

Generalized Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ,

- $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ a maximal chain,
- E a subset of the edges of $\mathcal{M}$

For $i \in \{0, \ldots, n-1\}$, $(\mathcal{M}, E)$ has a **descent** at index i if we have one of the following situations

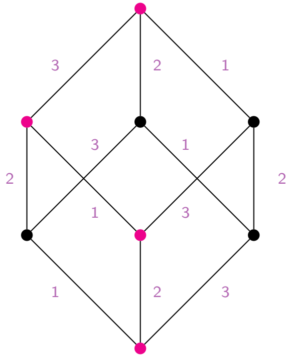


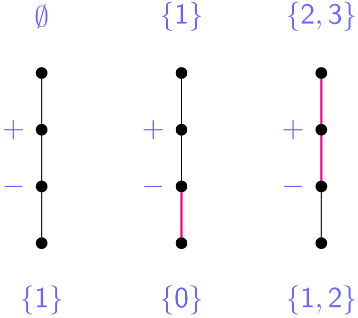
where $+$ means λ is increasing and $-$ means that λ is decreasing.

Now we include $i = 0$, which is a descent if and only if the edge above $\mathcal{M}_0$ is in E!

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





Generalized Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ,

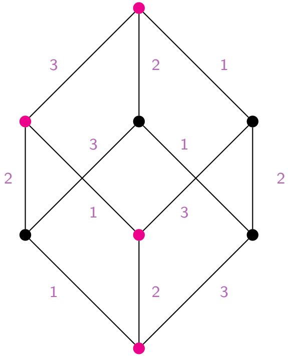
- $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ a maximal chain,
- E a subset of the edges of $\mathcal{M}$

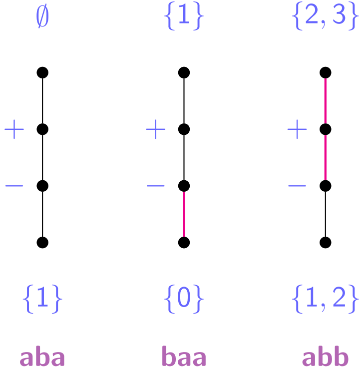
Then $\mathrm{mon}(M, E) = m_1 \ldots m_n$ is the monomial in noncommuting variables **a** and **b** with

$$m_i = \begin{cases} \mathbf{b} & \text{if } i \text{ is a descent of } (\mathcal{M}, E) \\ \mathbf{a} & \text{else}\,. \end{cases}$$

Generalized Descent Sets (Example)

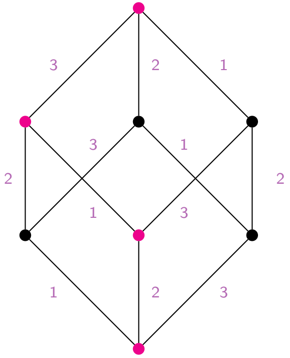
A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets and monomials for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.

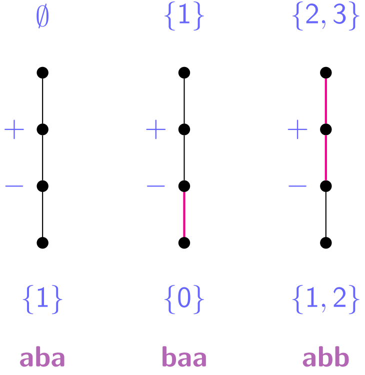




Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets and monomials for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





This descent statistic coincides with a statistic on *réseau* introduced by Bergeron, Mykytiuk, Sottile, and Willigenburg.

The coefficients of the extended **ab**-index

The Poincaré-extended **ab**-index

Let P be a graded poset.

Definition

The **extended ab-index** of P is

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then $\mathrm{ex}\Psi(P; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})$, and holds for all posets with R-labelings!

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2023)

The extended **ab**-*index of* P *is*

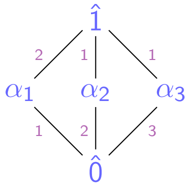
$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M}, E)} y^{\# E} \mathrm{mon}(\mathcal{M}, E)$$

where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chain and* E *is a subset of its edges.*

This immediately implies a Maglione–Voll's conjecture.

Example

Computing $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ using the theorem instead of the definition.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$$

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where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chains* E *is a subset of its edges.*

A Happy Surprise at $y = 1$
Unifying a Few Results from the (Ordinary) **ab**-index

The (ordinary) **ab**-index

Definition

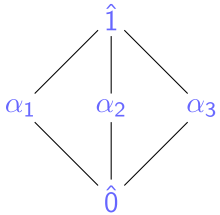
Let P be a graded poset. The **ab-index** of P is

$$\Psi(P;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P,\mathcal{C},0)\,\mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})\,.$$

Definition

Let P be a graded poset. The **ab-index** of P is

$$\Psi(P;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } P\backslash\{\hat{1}\}} \mathrm{Poin}(P,\mathcal{C},0)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})\,.$$



$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L},\mathcal{C};0)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+0+0$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+0$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0}<\alpha_i\}$	$(1+0)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\Psi(\mathcal{L};y,\mathbf{a},\mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2+\mathbf{b}(\mathbf{a}-\mathbf{b})\\ &\quad+3\cdot(\mathbf{a}-\mathbf{b})\mathbf{b}+3\mathbf{b}^2\\ &= \mathbf{a}^2+2\mathbf{ab}\end{aligned}$$

The ω-map

Definition

Let m be a monomial in **a** and **b**. Define a transformation ω that first sends **ab** to $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}$, then all remaining **a**'s to $\mathbf{a} + y\mathbf{b}$ and all remaining **b**'s to $\mathbf{b} + y\mathbf{a}$.

If $\mathrm{m} = \mathbf{aabba}$, then

$$\omega(\mathrm{m}) = (\mathbf{a} + y\mathbf{b})(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb})(\mathbf{b} + y\mathbf{a})(\mathbf{a} + y\mathbf{b}).$$

By extending ω linearly, we can apply this map to sums of monomials, i.e.,

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= (\mathbf{a} + y\mathbf{b})(\mathbf{a} + y\mathbf{b}) + 2(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}) \\ &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}\,.\end{aligned}$$

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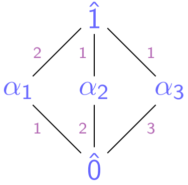
You might recognize these polynomials from earlier in this talk...

The ω-map

The **ab** index of the following poset is $\mathbf{aa} + 2\mathbf{ab}$.



We just saw that

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb} \\ &= \mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})\,.\end{aligned}$$

This is not a coincidence!

The ω-map

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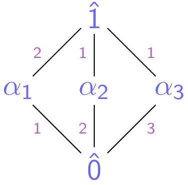

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$$\omega(\mathbf{aa} + 2\mathbf{ab}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}$$
$$= \mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})\,.$$

This is not a coincidence!

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

The ω-map

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

This recovers and refines and unifies several well-known results:

- When P is the lattice of flats of an *oriented matroid*, setting $y = 1$ recovers the ω map of Billera-Ehrenborg-Readdy,
- When P is the lattice of flats of an *oriented interval greedoid*, setting $y = 1$ recovers the ω map of Saliola-Thomas, and
- When P is a *distributive lattice*, setting $y = r + 1$ recovers the ω_r map of Ehrenborg (related to the "r-Signed Birkoff poset" from Hsiao).

The ω-map

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All three of these were proved with similar techniques, although no *unified* proof was known until now!

Take-aways

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

- It suffices to show that

$$\omega(\mathrm{mon}(\mathcal{M}, \emptyset)) = \sum_{E} y^{\#E} \mathrm{mon}(\mathcal{M}, E)$$

  for every maximal chain $\mathcal{M}$.
- Since the first letter of $\mathrm{mon}(\mathcal{M}, \emptyset)$ is always an $\mathbf{a}$, we can decompose $\mathrm{mon}(\mathcal{M}, \emptyset)$ into a product of monomials of the form $\mathbf{ab} \cdots \mathbf{b}$.

A few Questions

- There are posets not admitting R-labelings, which have nonnegative extended **ab**-indexes. What is this larger class of posets?
- What can we say about the coefficients of anayltic zeta functions themselves (these can have negative coefficients, but perhaps there are combinatorial interpretations)? What about the motivic zeta functions of JKU?
- The ω map can be reframed in terms of *peaks*. Setting $y = 1$ or $y = 0$ recovers well-studied combinatorics connected to *peak enumeration* and *quasisymmetric functions*. What can be said about y-refined peak enumerators?

Thank you!

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