The complexified complement of an arrangement–set pair

Galen Dorpalen-Barry

joint with Nick Proudfoot

SLMath Hot Topics Workshop:
Artin Groups and Arrangements
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UNIVERSITY OF OREGON

Outline

Setup & Motivation

The first kind of complement

- A **real hyperplane** is an affine linear subspace of codimension 1 in $V \cong \mathbb{R}^d$.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.

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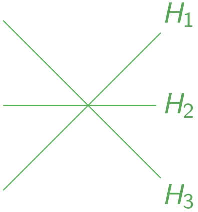
- A **chamber** of an arrangement is an open, connected component of

$$V \setminus \bigcup_{H \in \mathcal{A}} H\,.$$

This arrangement has 6 chambers.



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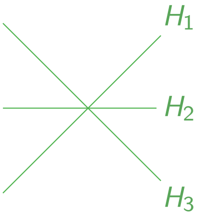


The only "interesting" topological data is the number or connected components.

Another kind of complement

- The **complexification** of a real vector space V is $V^{\mathbb{C}} = V + iV$.
- If H is a hyerplane defined as the zero set of f, its **complexification** $H^{\mathbb{C}}$ is the zero set of

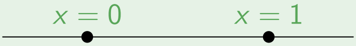
$$f^{\mathbb{C}} = f + i(f - f(0))\,.$$

- For a real arrangement $\mathcal{A}$, the **complexified complement** of $\mathcal{A}$ is

$$M(\mathcal{A}) = V^{\mathbb{C}} \setminus \bigcup_{H \in \mathcal{A}} H^{\mathbb{C}}\,.$$

Example

Consider the following arrangement of two hyperplanes in $\mathbb{R}$



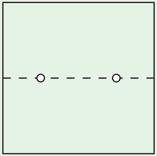
The complexified complement lives in $\mathbb{C}$ ...

Another kind of complement

Example

Consider the following arrangement of two hyperplanes in $\mathbb{R}$

The complexified complement is $\mathbb{C} \setminus \{z = 0, z = 1\}$ and looks like



The dashed line represents $\text{im}(z) = 0$ the two dots are the two punctures where $\text{im}(z) = 0$ and the real part lies on a hyperplane.

The complexified complement is topologically interesting

- The cohomology ring of $M(\mathcal{A})$ has a combinatorial presentation depending only on the underyling matroid of $\mathcal{A}$
  - Orlik-Brieskorn gave a combinatorial presentation of the *cohomology* of the complexified complement
  - This presentation (the Orlik–Solomon algebra) relies only on the underlying *matroid* of the arrangement
- There are combinatorial homotopy models for $M(\mathcal{A})$. Two important ones for this talk come from
  - The **Salvetti Complex** (in particular, work of Salvetti and Paris)
  - The **Bug-eyed model** - Proudfoot introduced a "bug-eyed model" for $M(\mathcal{A})$, and Perterson–Tosteson used it to give a geometric proof of a theorem of Hyde

Many more people in this room have studied $M(\mathcal{A})$ and proved wonderful things about it...

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- The cohomology ring is as a combinatorial presentation depending only on the underyling matroid matroid
  - Orlik-Brieskorn gave a combinatorial presentation of the *cohomology* of the complexified complement
  - This presentation (the Orlik–Solomon algebra) relies only on the underlying *matroid* of the arrangement
- There are combinatorial homotopy models for $M(\mathcal{A})$
  - Motivated by work of Deligne, Salvetti and then Paris described a simplicial complex, and showed that this is a homotopy model for complexified complements
  - The **Bug-eyed model** - Proudfoot introduced a "bug-eyed model" for $M(\mathcal{A})$, and Perterson–Tosteson used it to give a geometric proof of a theorem of Hyde

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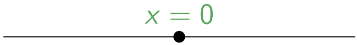
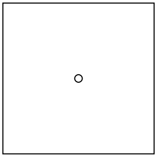
A bug-eyed model for $M(\mathcal{A})$

A motivating example

Consider the following arrangement of one hyperplane in $\mathbb{R}$



The complexified complement is $\mathbb{C} \setminus \{z = 0\}$ and looks like



Send $x + iy \in M(\mathcal{A})$ to a line with a double point $\{+, -\}$ at zero:

$$\begin{aligned} iy &\mapsto \operatorname{sign}(y) \\ x + iy &\mapsto x \qquad\qquad \text{if } x \neq 0\,. \end{aligned}$$

A motivating example

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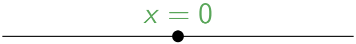

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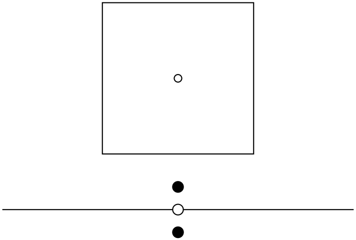


The fibers of this map all look like copies of $\mathbb{R}$.

A motivating example

Let $\mathbb{D}$ denote the line with a double point.

Theorem (Proudfoot '07, First Version)

The following action of $\mathbb{R}$ on $\mathbb{C}^*$ realizes $\mathbb{C}^*$ as a principle $\mathbb{R}$-bundle over $\mathbb{D}$:

$$\lambda * z = re(z) + i(e^{2\lambda}x^2 - e^{-2\lambda}y^2)$$

where $\lambda \in \mathbb{R}$, $z \in \mathbb{C}^*$, and $x + iy = \sqrt{-iz}$.

Key take-aways:

- $\mathbb{C}^*$ is a fiber bundle over $\mathbb{D}$ with fibers $\mathbb{R}$
- $\mathbb{C}^*$ is (weakly) homotopy equivalent to $\mathbb{D}$
- the real part of z is preserved under the action of $\mathbb{R}$, and the map to $\mathbb{D}$
- this extends to $(\mathbb{C}^*)^n$ and $\mathbb{D}^n$

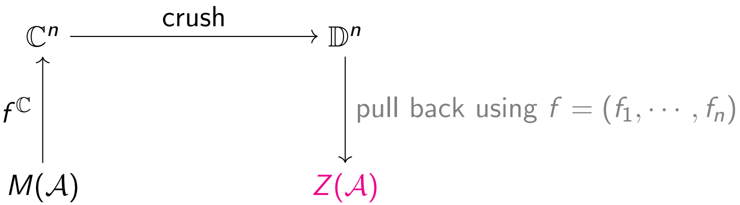
Using this idea for other arrangements

Observation

When $\mathcal{A} = \{H_1, \ldots, H_n\}$ is essential, the defining equations f_i of the H_i define an embedding of $M(\mathcal{A})$ into $\mathbb{C}^d$ via:

$$f^{\mathbb{C}} = (f_1^{\mathbb{C}}(z), \cdots, f_n^{\mathbb{C}}(z)) \in \mathbb{C}^n .$$

We'll like to define a space $Z(\mathcal{A})$ that is the image of $M(\mathcal{A})$ under a some maps that look like this:



Using this idea for other arrangements

On the next slide, we will describe $Z(\mathcal{A})$ combinatorially, with the plan to obtain the following theorem:

Theorem (Proudfoot '07, Second Version)

$M(\mathcal{A})$ is a fiber bundle over $Z(\mathcal{A})$, and the fibers are contractible.

Planned take-aways:

- $M(\mathcal{A})$ is (weakly) homotopy equivalent to $Z(\mathcal{A})$
- We have surjections

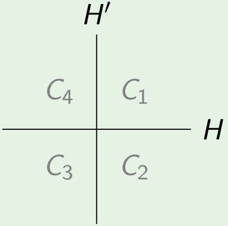
$$M(\mathcal{A}) \twoheadrightarrow Z(\mathcal{A}) \twoheadrightarrow V\,.$$

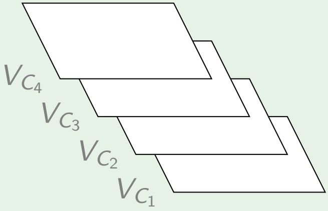
What is $Z(\mathcal{A})$?

Take one copy of V for each chamber of $\mathcal{A}$ and identify V_C with $V_{C'}$ along the complements of separating sets.

Example

Consider the following arrangement in $\mathbb{R}^2$:





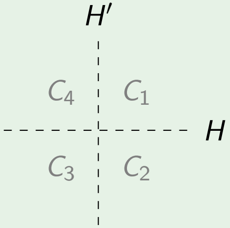
$V_{C_1} \cap V_{C_2}$ is $\mathbb{R}^2 \setminus \{H\}$ and $V_{C_1} \cap V_{C_3}$ is $\mathbb{R}^2 \setminus \{H, H'\}$

What is $Z(\mathcal{A})$?

Intuitively: replace all hyperplanes with "seams" corresponding to the positive and negative sides of each of the hyperplanes.

Example

For the boolean arrangement in $\mathbb{R}^2$, we have:



A point (x, y) in $Z(\mathcal{A})$ corresponds to a point in (x, y), together with some extra data about the hyperplanes containing (x, y).

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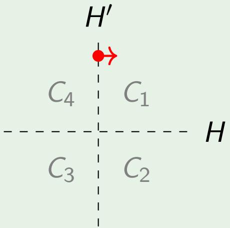

- H' separates $\{C_3, C_4\}$ from $\{C_1, C_2\}$
- In $Z(\mathcal{A})$ there are two copies of $(0, 2)$: the $\{C_3, C_4\}$ copy and the $\{C_1, C_2\}$ copy

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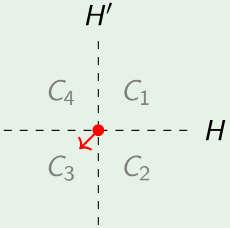

- H and H' separate all the chambers from one another
- In $Z(\mathcal{A})$ there are four copies of $(0, 0)$, corresponding to each of the four chambers

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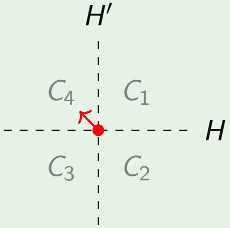

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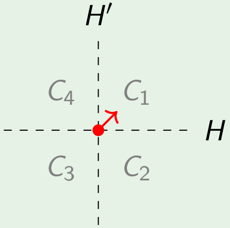

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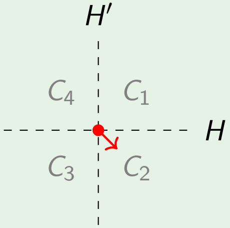

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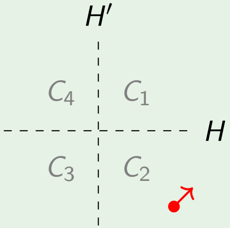

- Points that lie interior to chambers are contained in all V_C
- There's only one copy of each of those
- We still draw this little arrow, but it lies completely inside the chamber

$Z(\mathcal{A})$ is (weakly) homotopy equivalent to $M(\mathcal{A})$

Now we have the following:

Theorem (Proudfoot '07, Second Version)

$M(\mathcal{A})$ is a fiber bundle over $Z(\mathcal{A})$, and the fibers are contractible.

Note: $\mathcal{A}$ does not need to be essential!

Key take-aways:

- $M(\mathcal{A})$ is (weakly) homotopy equivalent to $Z(\mathcal{A})$
- We have surjections

$$M(\mathcal{A}) \twoheadrightarrow Z(\mathcal{A}) \twoheadrightarrow V\,.$$

Question:
Why is $Z(\mathcal{A})$ natural to consider from the perspective of the Salvetti poset?

The Salvetti Complex and a Nerve Lemma

Faces of Arrangements: Some Notation

Recall that an arrangement $\mathcal{A}$ decomposes V into relatively open cells called **faces**, which are intersections of the hyperplanes and halfspaces of the arrangement, i.e.,

$$\bigcap_{H \in \mathcal{A}} H^{\varepsilon_H} \qquad \text{for } \varepsilon_H \in \{0, +, -\}\,.$$

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We'll use the following notation:

- $\Sigma(\mathcal{A})$ is the set of faces
- $\mathcal{C}(\mathcal{A})$ is the set of chambers (= top-dimensional faces).
- $\Sigma(\mathcal{A})$ is a semigroup under $\circ$, where $F \circ G$ means "perturb F a little bit toward G" or equivalently, for each hyperplane: if ε_H is zero in F, use the choice of ε_H from G instead.

What is sal($\mathcal{A}$)?

The **Salvetti poset** is a poset on

$$\{(F,C) \in \mid F \in \Sigma(\mathcal{A}), C \in \mathcal{C}(\mathcal{A}), C = F \circ C\}$$

with the relation $(F,C) \prec (F',C')$ if and only if

$$F' = F \circ F' \qquad \text{and} \qquad C' = F' \circ C\,.$$

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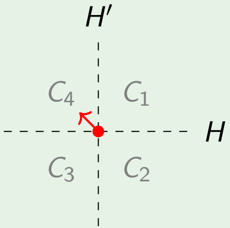

In sal($\mathcal{A}$) there are four ways to pair $(0,0)$ with an adjacent chamber.

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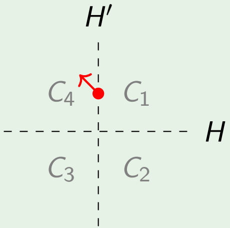

Moving up in the poset means sliding into a higher-dimensional face, and keeping the arrow pointed in the same direction.

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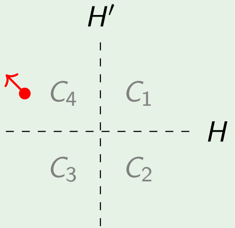

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By comparing these pictures with the ones we drew before, we see:

Remarkable Coincidence

The partial order on sal($\mathcal{A}$) is the containment order on connected components of intersections of V_C's in $Z(\mathcal{A})$.

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Want: Use a nerve lemma to say that the order complex of sal($\mathcal{A}$) is (weakly) homotopy equivalent to $Z(\mathcal{A})$.

Nerve Lemmas: a Typical Setup

- X is a topological space which is homeomorphic to a CW complex
- $\mathcal{U} = \{U \subseteq X\}$ is a finite open cover of X
- $\mathrm{Nerv}(\mathcal{U}) = \{\mathcal{V} \subseteq \mathcal{U} \mid \bigcap_{U \in \mathcal{V}} U \neq \emptyset\}$

Theorem (A Typical Nerve Lemma)

If arbitrary intersections of elements of $\mathcal{U}$ are empty or contractible, then X is homotopy equivalent to the order complex of $\mathrm{Nerv}(\mathcal{U})$ ordered by containment.

Example (Circle)

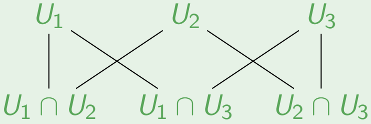
Cover the circle with three intervals

$$U_1 = (0, \pi) \qquad U_2 = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \qquad U_3 = \left(\frac{5\pi}{4}, \frac{\pi}{4}\right)$$

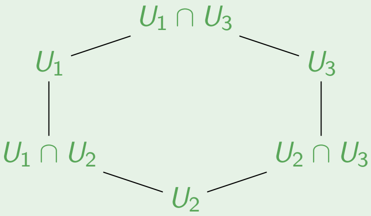
Nerve Lemmas: a Typical Setup

Example (Circle)

Our open cover of the circle has the following poset of intersections:



Which has the following order complex:



A Nerve Lemma for $Z(\mathcal{A})$?

$Z(\mathcal{A})$ has an natural open cover:

$$\mathcal{U} = \{V_C \mid C \in \mathcal{C}(\mathcal{A})\}$$

Problem: arbitrary intersections are not always contractible

Solution: we need a special version of the nerve lemma

A Nerve Lemma for $Z(\mathcal{A})$?

Setup:

- P: finite poset
- $Op(X)$: is the set of open subsets of X, ordered by inclusion
- $f : P \to Op(X)$ order-preserving with $f(p)$ contractible $\forall p \in P$
- For $x \in X$, define $P_x = \{p \in P \mid x \in f(P)\}$

We have the following generalization of the nerve lemma, due to Dugger:

Theorem

If the order complex of P_x is contractible for all $x \in X$, then X is weakly homotopy equivalent to the order complex of P.

A Nerve Lemma for $Z(\mathcal{A})$?

Theorem

If the order complex of P_x is contractible for all $x \in X$, then X is weakly homotopy equivalent to the order complex of P.

How we'll use it: Define a map $f : \text{sal}(\mathcal{A}) \to Op(X)$ by sending (F, C) to the connected component of

$$\bigcap_{C' : C' = F \circ C'} V_{C'} \qquad \text{containing } C.$$

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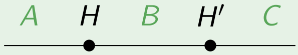

The pair (H, A) is sent to the open connected component of $V_A \cap V_B$ containing A.

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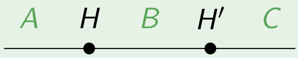
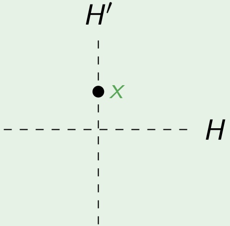


The pair (A, A) is sent to the open connected component of V_A containing A.

Nerve Lemma in Action

Example

Back to the Boolean arrangement in $\mathbb{R}^2$.



For a face F of $\mathcal{A}$, define

$$X_F = \bigcap_{C' : C' = F \circ C'} V_{C'} \, .$$

If $x \in X_F$, then lies in the right halfspace defined by H'.

If $x \in X_F$, then it lies in a unique connected component.

Nerve Lemma in Action

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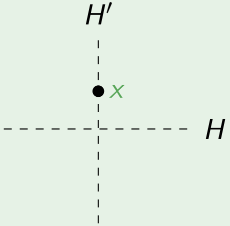

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If $x \in X_F$, then it lies in a unique connected component.

Observation: Contractibility of P_x is equivalent to the contractibility of the part of the face poset of $\mathcal{A}$ that cuts through a polyhedral cone.

A Nerve Lemma for $Z(\mathcal{A})$?

By looking at the face posets of pairs $(\mathcal{A}, \mathcal{K})$, we see that $\mathrm{sal}(\mathcal{A})$ and $Z(\mathcal{A})$ satisfy the conditions for Dan's Nerve lemma, and we obtain:

Theorem (DB-P 24+, First Version)

$\mathrm{sal}(\mathcal{A})$ *is (weakly) homotopy equivalent to* $Z(\mathcal{A})$

Comments:

- Part of this argument uses the machinery of *conditional oriented matroids*, and mildly extends work of Bandelt–Chepoi–Knauer.
- This recovers the well-known fact that $M(\mathcal{A})$ is (weakly) homotopy equivalent to $\mathrm{sal}(\mathcal{A})$.
- These arguments hold more generally.

Arrangement–Set Pairs

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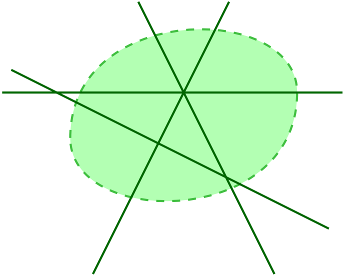
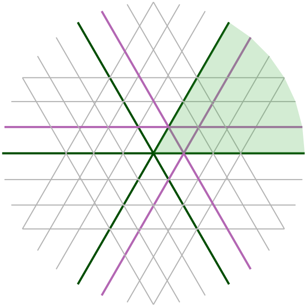
An **arrangement–set pair** is $(\mathcal{A}, \mathcal{K})$ where

1. $\mathcal{A}$ is an arrangement in $V \cong \mathbb{R}^d$
2. $\mathcal{K} \subseteq V$ is an open, convex set

We will be interested in the way the arrangement $\mathcal{A}$ subdivides $\mathcal{K}$.





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Example

Some examples where specific $(\mathcal{A}, \mathcal{K})$ pairs appear in the literature:

- **Braid Arrangement**: "cone–preposet dictionary" of Postnikov–Reiner–Williams, chambers count linear extensions
- **Catalan & Shi arrangements** associated to a finite crystallographic root system: symmetries among their **Weyl cones**
- **Alcoved polytopes**: convex unions of chambers of an affine Weyl arrangement of Type A_n, the subdivision by the hyperplanes gives a triangulation

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Features of a Good Answer:

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- $M(\mathcal{A}, \mathcal{K})$ and $Z(\mathcal{A}, \mathcal{K})$ are (weakly) homotopy equivalent
- There is a combinatorially-defined simplicial complex, generalizing the Salvetti complex, which is homotopy equivalent to $M(\mathcal{A}, \mathcal{K})$
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Weak homotopy equivalence for $M(\mathcal{A},\mathcal{K})$ and $Z(\mathcal{A},\mathcal{K})$

Recall the surjections from earlier

$$M(\mathcal{A}) \twoheadrightarrow Z(\mathcal{A}) \twoheadrightarrow V\,.$$

Restricting to $\mathcal{K}$ gives

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The first map is a restriction of a fiber bundle to subset, so it is again a fiber bundle. Using the results from before gives:

Proposition

$M(\mathcal{A}, \mathcal{K})$ and $Z(\mathcal{A}, \mathcal{K})$ are weakly homotopy equivalent.

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Theorem (DB-P 24+, Second Version)

$\mathrm{sal}(\mathcal{A},\mathcal{K})$ *is (weakly) homotopy equivalent to* $Z(\mathcal{A},\mathcal{K})$

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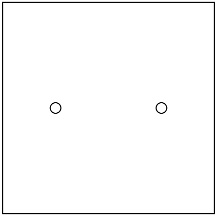
The last bullet is a topic for another talk.

An Extended Example

Extended Example

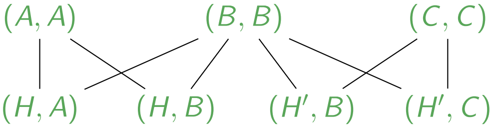
Consider the following arrangement of two hyperplanes in $\mathbb{R}$ with three chambers A, B, and C:

The complexified complement $M(\mathcal{A})$ is:

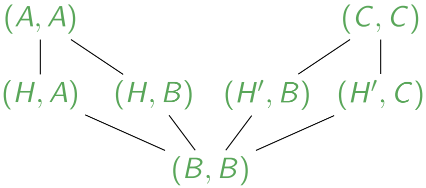


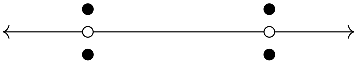
Extended Example

The Salvetti poset sal($\mathcal{A}$) is:



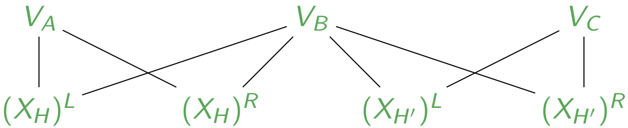
The order complex of sal($\mathcal{A}$) and $Z(\mathcal{A})$ are:



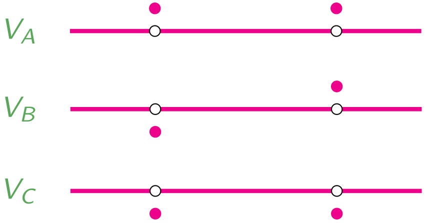


Extended Example

The image $\mathrm{sal}(\mathcal{A})$ in $Op(Z(\mathcal{A}))$ is



where the open top row of open sets are



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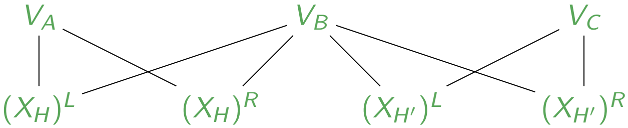
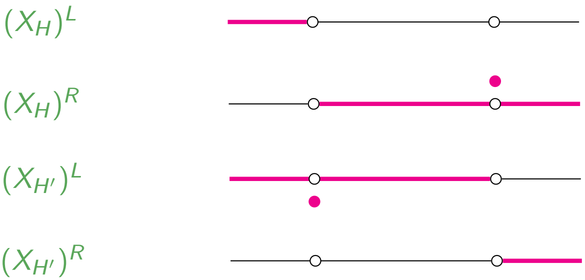


and the bottom row of open sets are



Thank you for listening!

Questions? Compliments?