The Poincaré-extended **ab**-index

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joint with Joshua Maglione and Christian Stump
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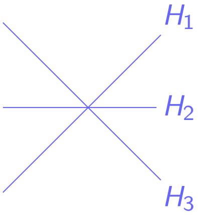
MFO Workshop:
Geometric, Algebraic, and Topological Combinatorics
April 7, 2024

Outline

Setup + Motivation

Arrangements of Hyperplanes in $\mathbb{R}^d$

- A **hyperplane** is an affine linear subspace of codimension 1.
- A collection of finitely-many (distinct) hyperplanes is an **arrangement**.



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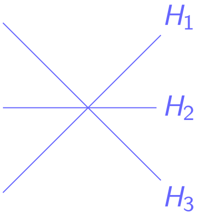

Today we'll focus on **intersections** (= nonempty intersections of some of the hyperplanes).

Arrangements of Hyperplanes in $\mathbb{R}^d$

The set of intersections of this arrangement is

$$\mathbb{R}^2, H_1, H_2, H_3, H_1 \cap H_2 \cap H_3$$

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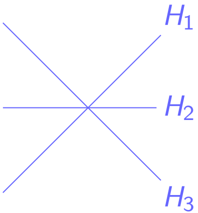
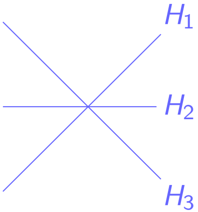


Today we'll focus on **intersections** (= nonempty intersections of some of the hyperplanes).

Poset of Intersections

Let $\mathcal{A}$ be an arrangement in $V \cong \mathbb{R}^d$ with intersections $\mathcal{L}(\mathcal{A})$.

- The elements of $\mathcal{L}(\mathcal{A})$ form a poset under reverse inclusion.
- A theorem of Zaslavsky relates the Möbius function values of **lower intervals** $[V, X] \subseteq \mathcal{L}(\mathcal{A})$ to the number of **regions** of the arrangement.



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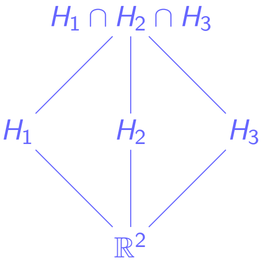

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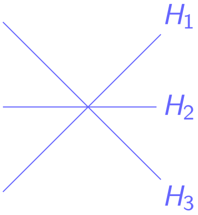

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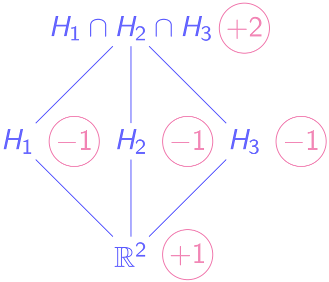

The Poincaré Polynomial of a Poset

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$$\mathrm{Poin}(\mathcal{L}; y) = \sum_{x \in \mathcal{L}} |\mu(\hat{0}, x)| \; y^{\mathrm{codim}(x)},$$

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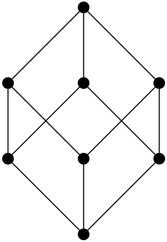
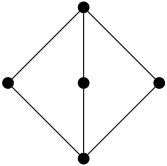
Tells us the Hilbert series of the Orlik-Solomon Algebra and Varchenko-Gelfand ring.



$1 + 3y + 3y^2 + y^3$



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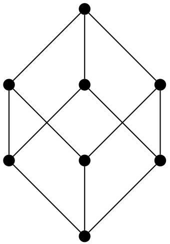
Note. We can define the Poincaré polynomial for any *graded poset*.



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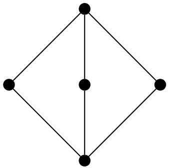

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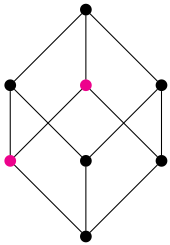
Chain Poincaré Polynomials

Let $\mathcal{A}$ be a central, essential hyperplane arrangement and $\mathcal{L}$ its lattice of intersections.

Moreover, let $\mathcal{C} = \{C_1 < \cdots < C_k\}$ be a chain of $\mathcal{L}$, i.e., a subset of the intersections which is totally ordered by inclusion.

The **chain Poincaré polynomial** of $\mathcal{C}$ is

$$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y) = \prod_{i=1}^{k} \mathrm{Poin}([C_i, C_{i+1}], y) \qquad \text{where } C_{k+1} = \hat{1}.$$



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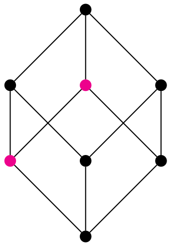

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Setting $y = 1$ recovers the size of a fiber of a chain under the *support map* $z : \Sigma^*(\mathcal{A}) \to \mathcal{L}$.

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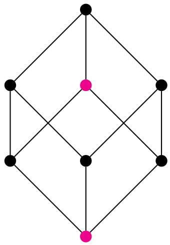

$$\mathrm{Poin}(P, \mathcal{C}; y) = (1 + 2y + y^2)(1 + y)$$

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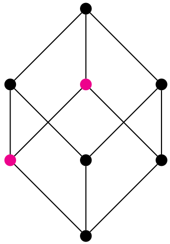
The Weight of a Chain

Let $\mathcal{A}$ be a central, essential hyperplane arrangement and $\mathcal{L}$ its lattice of intersections and let $\mathcal{C} = \{C_1 < \cdots < C_k\}$ be a chain of $\mathcal{L}$.

If P is rank n (every maximal chain from $\hat{0}$ to $\hat{1}$ has length $n+1$) then the **weight** of a chain $\mathcal{C}$ is $\text{wt}(\mathcal{C}) = w_1 \dots w_n \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ where

$$w_i = \begin{cases} \mathbf{b} & \text{if } \exists C_j \in \mathcal{C} \text{ such that } \text{rank}(C_j) = i-1 \\ \mathbf{a} - \mathbf{b} & \text{else}. \end{cases}$$



$\text{wt}(\mathcal{C}) = (\mathbf{a} - \mathbf{b})\mathbf{bb}$

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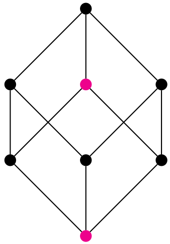

$$\mathrm{wt}(\mathcal{C}) = \mathbf{b}(\mathbf{a}-\mathbf{b})\mathbf{b}$$

The Poincaré-extended **ab**-index

Definition

The **(Poincaré-)extended ab-index** of $\mathcal{L}$ is

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } \mathcal{L}\setminus\{\hat{1}\}} \mathrm{Poin}(\mathcal{L}, \mathcal{C}, y)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

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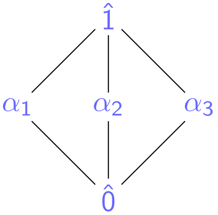

$\mathcal{C}$	$\mathrm{Poin}(\mathcal{L}, \mathcal{C}; y)$	$\mathrm{rank}(\mathcal{C})$	$\mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+3y+2y^2$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+y$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+y)^2$	$\{0,1\}$	$\mathbf{b}^2$

$$\begin{aligned}\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + (1+3y+2y^2)\mathbf{b}(\mathbf{a}-\mathbf{b}) + 3\cdot(1+y)(\mathbf{a}-\mathbf{b})\mathbf{b} + 3\cdot(1+y)^2\mathbf{b}^2 \\ &= \mathbf{a}^2 + (3y+2y^2)\mathbf{ba} + (2+3y)\mathbf{ab} + y^2\mathbf{b}^2\end{aligned}$$

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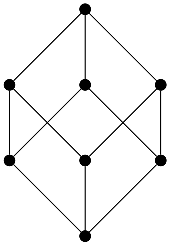

For the poset on the left:

$$\begin{aligned}\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \mathbf{a}^3 &+ (3y+2)\mathbf{a}^2\mathbf{b} + (3y^2+6y+2)\mathbf{aba} \\ &+ (3y^2+3y+1)\mathbf{ab}^2 + (y^3+3y^2+3y)\mathbf{ba}^2 \\ &+ (2y^3+6y^2+3y)\mathbf{bab} + (2y^3+3y^2)\mathbf{b}^2\mathbf{a} \\ &+ y^3\mathbf{b}^3\,.\end{aligned}$$

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Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(\mathcal{L}; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets!

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Their conjecture is true, even for $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$, and holds for an even bigger class of posets! Before we get into the proof, let's look at where their conjecture comes from...

Motivation: Analytic Zeta Functions

Let $\mathcal{A}$ be a central hyperplane arrangement in a real vector space with intersection lattice $\mathcal{L}$.

Maglione–Voll prove that (after a change of variables) the **(coarse) analytic zeta function** of $\mathcal{A}$ is

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C} \cup \{\hat{0}\}, y) \left(\frac{t}{1-t}\right)^{\#\mathcal{C}}.$$

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This is a bivariate version of the **analytic zeta function**.

A different bivariate specialization of their analytic zeta function recovers the celebrated **Motivic Zeta function** of a matroid given by Jensen–Kutler–Usatine.

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Putting all terms over the same denominator gives

$$Z_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \frac{\mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}}{(1-t)^{\mathrm{rank}(\mathcal{A})}}.$$

The numerator of this rational function is

$$Num_{\mathcal{A}}(y,t) = \sum_{\mathcal{C}:\text{chain of}\mathcal{L}\setminus\{\hat{0},\hat{1}\}} \mathrm{Poin}(\mathcal{C}\cup\{\hat{0}\},y)t^{\#\mathcal{C}}(1-t)^{\mathrm{rank}(\mathcal{A})-\#\mathcal{C}}.$$

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We can now state Maglione–Voll's conjecture more precisely:

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

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$Num_{\mathcal{A}}(y,t)$ has nonnegative coefficients.

Kühne–Maglione studied $Num_{\mathcal{A}}(1,t)$ as well, and conjectured that

$$\text{Poin}(\mathcal{A},1)\cdot(1+t)^{\text{rank}\mathcal{A}-1} \leq Num_{\mathcal{A}}(1,t).$$

We won't discuss it today, but our proof of the Maglione-Voll conjecture will give a proof of Kühne–Maglione's conjecture (almost) for free!

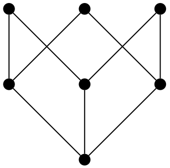
Combinatorial Machinery: R-labeled Posets + Generalized Descent Sets

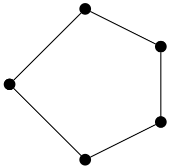
Graded Posets

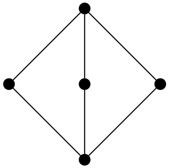
Let P be a poset with $\hat{0}$ and $\hat{1}$.

- A **chain** is a subset of the ground set which is totally ordered with respect to P.
- A chain $\mathcal{C} = C_1 < C_2 < \cdots C_n$ is **maximal** if C_i covers C_{i+1} for all $i = 1, \ldots, n-1$.
- P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.
- For $x, y \in P$, the **interval** between x and y is

$$[x, y] = \{z \mid x \leq z \leq y\}.$$





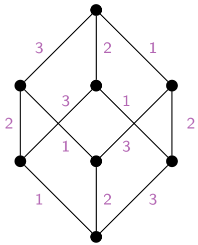


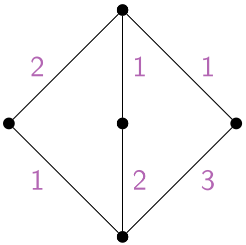
R-labelings

Let P be a graded poset, and let $\mathcal{E}(P) = \{(x, y) \mid x, y \in P,\ x \lessdot y\}$ denote the set of cover relations of P.

A labeling $\lambda : \mathcal{E}(P) \to \mathbb{Z}$ is an R-**labeling** if for every interval $[x, y]$, there is a unique maximal chain $\mathcal{M} = \{x = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = y\}$ such that the labels *weakly* increase, i.e.,

$$\lambda(C_{i-1}, C_i) \leq \lambda(C_i, C_{i+1}) \qquad \text{for } i = 2, \ldots k - 1.$$



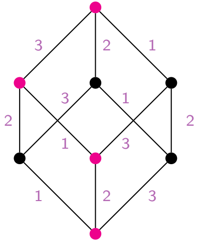


Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ.

Let $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ be a maximal chain of P. For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1}).$$



This chain has a descent at position 1.

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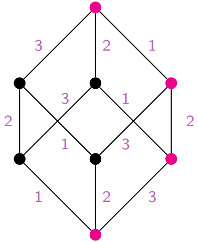
For $i \in \{1, \ldots, n-1\}$, $\mathcal{M}$ has a **descent** at index i if

$$\lambda(C_{i-1}, C_i) > \lambda(C_i, C_{i+1})\,.$$



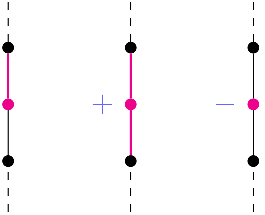
This chain has descents at positions 1 and 2.

Generalized Descent Sets

Let P be a graded poset of rank n, with a fixed R-labeling λ,

- $\mathcal{M} = \{\hat{0} = C_0 \lessdot C_1 \lessdot \cdots \lessdot C_{k-1} \lessdot C_k = \hat{1}\}$ a maximal chain,
- E a subset of the edges of $\mathcal{M}$

For $i \in \{0, \ldots, n-1\}$, $(\mathcal{M}, E)$ has a **descent** at index i if we have one of the following situations

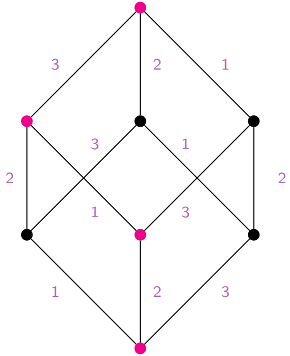


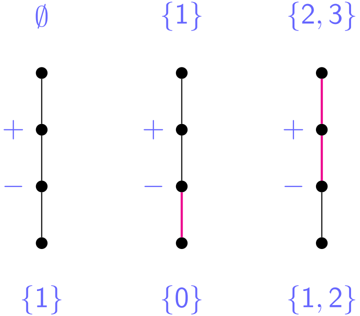
where $+$ means λ is increasing and $-$ means that λ is decreasing.

Now we include $i = 0$, which is a descent if and only if the edge above $\mathcal{M}_0$ is in E!

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





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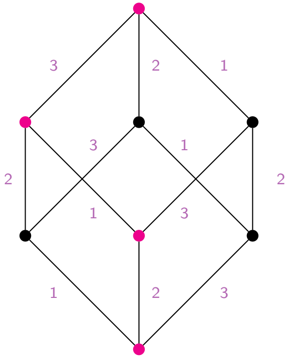
- E a subset of the edges of $\mathcal{M}$

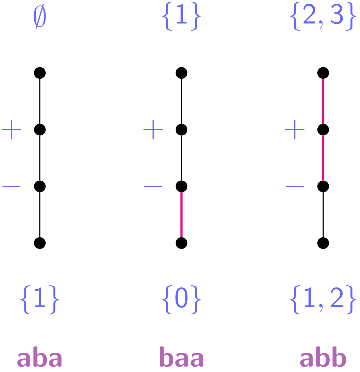
Then $\text{mon}(M, E) = m_1 \ldots m_n$ is the monomial in noncommuting variables **a** and **b** with

$$m_i = \begin{cases} \mathbf{b} & \text{if } i \text{ is a descent of } (\mathcal{M}, E) \\ \mathbf{a} & \text{else}. \end{cases}$$

Generalized Descent Sets (Example)

A maximal chain $\mathcal{M}$ in an R-labeled poset, together with the descent sets and monomials for the $(\mathcal{M}, E)$ pairs with $E = \emptyset,\ \{1\},\ \{2,3\}$.





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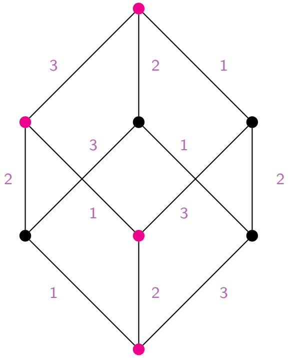
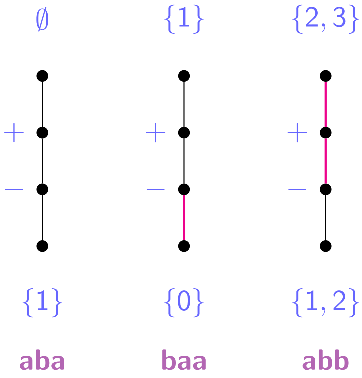




This descent statistic coincides with a statistic on *réseau* introduced by Bergeron, Mykytiuk, Sottile, and Willigenburg.

The coefficients of the extended **ab**-index

The Poincaré-extended **ab**-index

Let P be a graded poset.

Definition

The **extended ab-index** of P is

$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}\text{:chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P, \mathcal{C}, y)\, \mathrm{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$

Conjecture (Maglione-Voll)

If P is the *intersection poset* of an arrangement of hyperplanes, then (a harmless modification of) $\mathrm{ex}\Psi(P; y, 1, t)$ has nonnegative coefficients.

Their conjecture is true, even for $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b})$, and holds for all posets with R-labelings!

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2023)

The extended **ab***-index of P is*

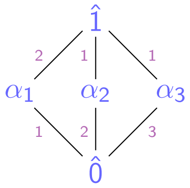
$$\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \sum_{(\mathcal{M}, E)} y^{\#E} \mathrm{mon}(\mathcal{M}, E)$$

where the sum ranges over all pairs $(\mathcal{M}, E)$ where $\mathcal{M}$ is a maximal chain and E is a subset of its edges.

This immediately implies a Maglione–Voll's conjecture.

Example

Computing $\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b})$ using the theorem instead of the definition.



E	$y^{\#E}$	$\hat{0} \lessdot \alpha_1 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_2 \lessdot \hat{1}$	$\hat{0} \lessdot \alpha_3 \lessdot \hat{1}$
$\{\}$	1	**aa**	**ab**	**ab**
$\{1\}$	y	**ba**	**ba**	**ba**
$\{2\}$	y	**ab**	**ab**	**ab**
$\{1,2\}$	y^2	**bb**	**ba**	**ba**

$$\mathrm{ex}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) = \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (2 + 3y)\mathbf{ab} + y^2\mathbf{bb}$$

The Poincaré-extended **ab**-index

Let P be a graded poset of rank n with an R-labeling λ.

Theorem ((DB)MS, 2023)

The extended **ab***-index of* P *is*

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where the sum ranges over all pairs $(\mathcal{M}, E)$ *where* $\mathcal{M}$ *is a maximal chains* E *is a subset of its edges.*

Let's look at a short sketch of the proof...

(Overly-Simplified!) Proof Outline

Let P be a graded poset of rank n with an R-labeling λ.

Step 1: Use the following theorem to reinterpret the chain Poincaré polynomial as a sum over maximal chains with certain increasing-decreasing pattern with respect to the R-labeling.

Theorem

Let P be a poset with R-labeling λ. For $x, y \in P$ with $x < y$, we have

$$(-1)^{\text{rank}(x,y)}\mu(x,y) = \#\{\textit{decreasing maximal chains in } [x,y]\}.$$

Step 2: Use inclusion-exclusion to describe the coefficients as sets.

Step 3: Show that the elements at the top of this inclusion-exclusion argument are in bijection with pairs $(\mathcal{M}, E)$.

A few Questions

- There are posets not admitting R-labelings, which have nonnegative extended **ab**-indexes. What is this larger class of posets?
- What can we say about the coefficients of anayltic zeta functions themselves (these can have negative coefficients, but perhaps there are combinatorial interpretations)? What about the motivic zeta functions of JKU?
- The ω map can be reframed in terms of *peaks*. Setting $y = 1$ or $y = 0$ recovers well-studied combinatorics connected to *peak enumeration* and *quasisymmetric functions*. What can be said about y-refined peak enumerators?

Danke!

Selected References

Louis J. Billera, Richard Ehrenborg, and Margaret Readdy.
The **c**-2**d**-index of oriented matroids.
J. Combin. Theory Ser. A, 80(1):79–105, 1997.

Lukas Kühne and Joshua Maglione.
On the geometry of flag Hilbert-Poincaré series for matroids.
Algebraic Combinatorics (to appear), 2023.

Joshua Maglione and Christopher Voll.
Flag Hilbert–Poincaré series of hyperplane arrangements and Igusa zeta functions.
Israel Journal of Mathematics (to appear), 2023.

Connection to the (ordinary) **ab**-index

The (ordinary) **ab**-index

Definition

Let P be a graded poset. The **ab-index** of P is

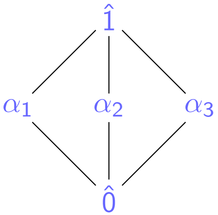
$$\Psi(P;\mathbf{a},\mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } P\setminus\{\hat{1}\}} \mathrm{Poin}(P,\mathcal{C},0)\ \mathrm{wt}_{\mathcal{C}}(\mathbf{a},\mathbf{b})\,.$$

The (ordinary) **ab**-index

Definition

Let P be a graded poset. The **ab-index** of P is

$$\Psi(P; \mathbf{a}, \mathbf{b}) = \sum_{\mathcal{C}:\text{chain of } P \setminus \{\hat{1}\}} \text{Poin}(P, \mathcal{C}, 0)\ \text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})\,.$$



$\mathcal{C}$	$\text{Poin}(\mathcal{L}, \mathcal{C}; 0)$	$\text{rank}(\mathcal{C})$	$\text{wt}_{\mathcal{C}}(\mathbf{a}, \mathbf{b})$
$\{\}$	1	$\{\}$	$(\mathbf{a}-\mathbf{b})^2$
$\{\hat{0}\}$	$1+0+0$	$\{0\}$	$\mathbf{b}(\mathbf{a}-\mathbf{b})$
$\{\alpha_i\}$	$1+0$	$\{1\}$	$(\mathbf{a}-\mathbf{b})\mathbf{b}$
$\{\hat{0} < \alpha_i\}$	$(1+0)^2$	$\{0, 1\}$	$\mathbf{b}^2$

$$\begin{aligned}\Psi(\mathcal{L}; y, \mathbf{a}, \mathbf{b}) &= (\mathbf{a}-\mathbf{b})^2 + \mathbf{b}(\mathbf{a}-\mathbf{b}) \\ &\quad + 3 \cdot (\mathbf{a}-\mathbf{b})\mathbf{b} + 3\mathbf{b}^2 \\ &= \mathbf{a}^2 + 2\mathbf{ab}\end{aligned}$$

The ω-map

Definition

Let m be a monomial in **a** and **b**. Define a transformation ω that first sends **ab** to $\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}$, then all remaining **a**'s to $\mathbf{a} + y\mathbf{b}$ and all remaining **b**'s to $\mathbf{b} + y\mathbf{a}$.

If m = **aabba**, then

$$\omega(\mathrm{m}) = (\mathbf{a} + y\mathbf{b})(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb})(\mathbf{b} + y\mathbf{a})(\mathbf{a} + y\mathbf{b}).$$

By extending ω linearly, we can apply this map to sums of monomials, i.e.,

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= (\mathbf{a} + y\mathbf{b})(\mathbf{a} + y\mathbf{b}) + 2(\mathbf{ab} + y\mathbf{ba} + y\mathbf{ab} + y^2\mathbf{bb}) \\ &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb}\,.\end{aligned}$$

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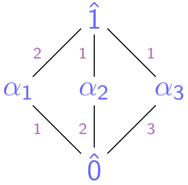
You might recognize these polynomials from earlier in this talk...

The ω-map

The **ab** index of the following poset is $\mathbf{aa} + 2\mathbf{ab}$.



We just saw that

$$\begin{aligned}\omega(\mathbf{aa} + 2\mathbf{ab}) &= \mathbf{aa} + (3y + 2y^2)\mathbf{ba} + (3y + 2)\mathbf{ab} + y^2\mathbf{bb} \\ &= \mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) .\end{aligned}$$

This is not a coincidence!

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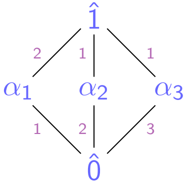

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This is not a coincidence!

Theorem ((DB)MS, 2023)

For an R-labeled poset P, we have $\mathrm{ex}\Psi(P; y, \mathbf{a}, \mathbf{b}) = \omega(\Psi(P; \mathbf{a}, \mathbf{b}))$.

The ω-map

Several specializations of the ω map have already appeared in the literature:

- When P is the lattice of flats of an *oriented matroid*, setting $y = 1$ recovers the ω map of Billera-Ehrenborg-Readdy,
- When P is the lattice of flats of an *oriented interval greedoid*, setting $y = 1$ recovers the ω map of Saliola-Thomas, and
- When P is a *distributive lattice*, setting $y = r + 1$ recovers the ω_r map of Ehrenborg (related to the "r-Signed Birkoff poset" from Hsiao).

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All three of these come from a pair of posets P, Q with an order- and rank- preserving surjection $z : P \to Q$ with the property that the size of the fiber $\#z^{-1}(\mathcal{C})$ of a chain $\mathcal{C}$ is an evaluation of $\mathrm{Poin}(Q, \mathcal{C}, y)$.